Assignment 3: Logical Agents: Solutions

Due 2/22 at 11:59pm.
Please work individually and use the submission instructions.

Q1. Basics (2 points)
State if each of the sentences below are unsatisfiable, satisfiable, or valid.

- $P \Rightarrow Q \quad \text{Satisfiable}$
- $P \Rightarrow \neg Q \quad \text{Satisfiable}$
- $P \Rightarrow P \quad \text{Valid}$
- $P \iff \neg P \quad \text{Unsatisfiable}$
- $P \Rightarrow (Q \Rightarrow P) \quad \text{Valid}$

Q2. Manipulating propositional sentences (3 points)
Show that any CNF formula (conjunction of clauses, where each clause is a disjunct of literals) can be converted to a 3-CNF formula (conjunction of clauses where each clause is at most a disjunction of 3 literals) where the 3-CNF formula is true in the same models that the original CNF sentence was true. Extra Credit (1 point). Show that a CNF formula can be converted to a 3-CNF formula where each clause has exactly 3 disjuncts.

ANSWER:
The following procedure converts a formula in CNF into 3-CNF with exactly 3 disjuncts. If we remove Step 1 from the procedure below, we get a 3-CNF form where each clause has at most 3 disjuncts.

1. In each clause which currently has one or two literals, we replicate one of the literals until the total number is three.

2. In each clause that has more than three literals we split it into several clauses and add several variables to preserve the satisfiability or nonsatisfiability of the original.
   a. For example, we replace clause $(a_1 \lor a_2 \lor a_3 \lor a_4)$ with $(a_1 \lor a_2 \lor z) \land (\neg z \lor a_3 \lor a_4)$ where $z$ is a new variable. If some assignment of the $a$'s satisfies the original clause, we can find some assignment of $z$ so that the two new clauses are satisfied.

3. In general, if the clause contains $k$ literals, $(a_1 \lor a_2 \lor \ldots \lor a_k)$, we can replace it with the $k$-2 clauses,
   a. $(a_1 \lor a_2 \lor z_1) \land (\neg z_1 \lor a_3 \lor z_2) \land (\neg z_2 \lor a_4 \lor z_3) \land \ldots \land (\neg z_{k-3} \lor a_{k-1} \lor a_k)$. 

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We may easily verify that the new formula is satisfiable if and only if the original formula was (the reader is invited to do so).

Q3. Propositional Resolution (2 points)

- Consider the sentence Heads, I win. Tails, you lose. Design a propositional KB that represents the sentence (create the propositions and rules required). Then use propositional resolution to prove that I always win.

ANSWER:

1. Make our objects:
2. State your rules:
   a. H ⇒ I and T ⇒ ¬ Y
3. Is that all? Don’t forget, you must specify implicit rules, too! The system doesn’t know that heads and tails are mutually exclusive.
   a. H ⊗ T and I ⊗ Y
4. Convert to CNF
   a. ¬ H ∨ I ¬ T ∨ ¬ Y (H ∨ T) ∧ (¬ H ∨ ¬ T)
   b. (I ∨ Y) ∧ (¬ I ∨ ¬ Y)
5. We want to prove I, so insert the literal ¬ I for the proof by contradiction. Now start resolving clauses:
   a. ¬ T ∨ ¬ Y and H ∨ T resolves to H ⊕ ¬ Y
   b. ¬ H ∨ I and H ∨ ¬ Y resolves to I ⊕ ¬ Y
   c. ¬ I and I ∨ ¬ Y resolves to ¬ Y
   d. I ∨ Y and ¬ Y resolves to I
   e. I and ¬ I {} -- we have a contradiction, so I is true.

Q4. Propositional Puzzle (3 points)

Write out the facts as sentences in Propositional Logic, and use propositional resolution to solve the crime.

1. There are three suspects for a murder: Adams, Brown, and Clark.
2. Adams says “I didn’t do it. The victim was old acquaintance of Brown’s. But Clark hated him.”
3. Brown states “I didn’t do it. I didn’t know the guy. Besides I was out of town all the week.”
4. Clark says “I didn’t do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it.”
5. Assume that the two innocent men are telling the truth, but that the guilty man might not be.

ANSWER:
The key to the solution is to make the suspect’s statements conditional on their innocence. The following rules/axioms are sufficient.

- Let Adams, Brown, and Clark be A, B, and C respectively. Let V be the victim.
- Let I (A) be the proposition that A is innocent, I (B) the proposition that B is innocent, and I (C) the proposition that C is innocent.
- Let F (A, V) indicate that A is a friend (acquaintance) of V and L (A,V) indicate that A liked V.
- Let W (A,V) be the proposition that A was with V on the day of the murder and W (B,V) the proposition that B was with V on that day.
- Let T (B) be the proposition that B was in town on the day of the murder.
- Let K (B,V) be the proposition that B knows V.

Here are the rules.

1. If A was innocent, then B was V’s friend and C did not like V, just as A said.
   1.1. I(A) ⇒ F (B, V)
   1.2. I(A) ⇒ ¬L (C, V)
2. If B was innocent, then he wasn’t in town and he doesn’t know V
   2.1. I(B) ⇒ ¬T(B)
   2.2. I(B) ⇒ ¬K(B, V)
3. If C was innocent, then A was with V and B was with V
   3.1. I(C) ⇒ W (A,V)
   3.2. I(C) ⇒ W (B,V)
4. Now some general knowledge rules in our KB
   4.1. W (B,V) ⇒ T (B)
   4.2. F (B,V) ⇒ K (B,V)
   4.3. L (C,V) ⇒ K (C,V)
5. Now we assert that only one of A, B, and C is the guilty person
   5.1. I (A) ∨ I (B)
   5.2. I (A) ∨ I (C)
   5.3. I (B) ∨ I (C)

Converting to CNF form leads to the first twelve clauses shown below. Let us say we want to prove that ¬I (B) (B did it).

We assert the negation I (B) and perform resolution. This is the clause 13 below

1. {¬I (A), F (B, V)}
2. {¬I (A), ¬L (C, V)}
3. {¬I (B), ¬T (B)}

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1 Of course these are first order statements of the form ∀x,y L(x, y) ⇒ K(x, y), etc. But for the purpose of this puzzle, we can instantiate them with the specific objects (A, B, C) to make them propositional and use propositional resolution to solve the puzzle.
4. \{\neg I (B), \neg K (B, V)\}
5. \{\neg I (C), W (A, V)\}
6. \{\neg I (C), W (B, V)\}
7. \{\neg W (B, V), T (B)\}
8. \{\neg F (B, V), K (B, V)\}
9. \{\neg L (C, V), K (C, V)\}
10. \{I (A), I (B)\}
11. \{I (A), I (C)\}
12. \{I (B), I (C)\}
13. \{I (B)\}
14. \{\neg I (A), K (B, V)\} -- RESOLVING Clauses 1 and 8
15. \{\neg I (C), T (B)\} -- RESOLVING Clauses 6 and 7
16. \{\neg I (A), \neg I (B)\} -- RESOLVING 4 and 14
17. \{\neg I (C), \neg I (B)\} -- RESOLVING 3 and 15
18. \{I (C), \neg I (B)\} -- RESOLVING 11 and 16
19. \{\neg I (B)\} -- RESOLVING 17 and 18
20. \{\}\ -- RESOLVING 13 and 19.

So B did it.

Q5. First Order Logic (Translation for FOL to Natural Language) (2 points)

Translate the following FOL sentences into colloquial English. You may assume the obvious meanings of all constants.

1. \forall x \ Hesitates(x) \Rightarrow Lost (x)
   Anyone (well, He) who hesitates is lost.
2. \neg \exists x \ Business(x) \land Like(x, Showbusiness)
   There is no business like Showbusiness.
3. \neg \forall x \ Glitters (x) \Rightarrow Gold(x)
   Not everything that glitters is gold.
4. \exists x \ Mushroom(x) \land (\forall z \neg (z = x) \Rightarrow \neg \text{Mushroom} (z))
   There is exactly one mushroom.

Q6. First Order Logic (Translation from Natural Language) (3 points)

Translate the following sentences. Use the following vocabulary.

- Male (x) means that the object denoted by x is male.
- Female (x) means that the object denoted by x is female.
- Vegetarian (x) means that x is vegetarian.
- Butcher (x) means that x is a butcher.
- Like (x, y) means that x likes y.

1. No man is both a butcher and a vegetarian.
   \neg \exists x \ Male (x) \land \text{Butcher} (x) \land \text{Vegetarian} (x)
2. All men except butchers like vegetarians.
∀x∀y Male (x) ∧ ¬ Butcher (x) ∧ Vegetarian (y) ⇒ Likes (x, y)

3. The only vegetarian butchers are women.
   ∀x Vegetarian (x) ∧ Butcher (x) ⇒ Female (x)

4. No man likes a woman who is vegetarian.
   ¬∃x∃y Male (x) ∧ Female (y) ∧ Vegetarian (y) ∧ Likes (x, y)

5. No woman likes a man who does not like all vegetarians.
   ¬∃x∃y∃z Female (x) ∧ Male (y) ∧ Vegetarian (z) ∧ ¬ Likes (y, z) ∧ Likes (x, y)