Q1. Review: Conditional Independence Assertions

Consider the following pairs of Bayes’ nets. If the two networks have identical conditional independences, write same, along with writing one of their shared independence (or none if they assert none). If the two networks have different conditional independences, write different, along with writing an independence that one has but not the other. For example, in the following case you would answer as shown:

\[
\text{different, right has } A \independent B | \{\}. \\
\]

(a)

Different, left has \( A \independent B | \{\} \), right has \( A \independent B | \{C\} \)

(b)

Same, both have \( A \independent B | \{C\} \)

(c)

Same, none.

(d)

Different. Left has \( \{A, B\} \independent D | \{C\} \), right has \( A \independent \{B, D\} | \{\} \), \( B \independent A, D | \{\} \), \( D \independent A, B | \{\} \)
Q2. Basic Inference

The next parts involve computing various quantities in the network below. These questions are designed so that they can be answered with a minimum of computation. If you find yourself doing a copious amount of computation for each part, step back and consider whether there is simpler way to deduce the answer.

(a) \( P(a, \neg b, c, \neg d) \)

\[
P(a) P(\neg b|a) P(c|a) P(\neg d|\neg b) = 0.1 \times 0.5 \times 0.4 \times 0.8 = 0.016
\]

(b) \( P(b) \)

\[
P(b) = \sum_{A=\{a, \neg a\}} P(A) P(b|A) = 0.1 \times 0.5 + 0.9 \times 0.8 = 0.77
\]

(c) \( P(a|b) \)

\[
P(a|b) = \frac{P(a,b)}{P(b)} = \frac{P(a)P(b|a)}{P(b)} = \frac{0.1 \times 0.5}{0.77} = 0.064935
\]

(d) \( P(d|a) \)

\[
P(d|a) = \sum_{B=\{b, \neg b\}} P(d|B)p(B|a) = 0.9 \times 0.5 + 0.2 \times 0.5 = 0.55
\]

(e) \( P(d|a, c) \)

From the conditional independence properties of the graph, \( D \perp \perp C|\{A\} \). Hence, \( P(d|a, c) = p(d|a) = 0.55 \)
Q3. Elimination

Consider the following Bayes net,

and suppose we want to compute \( P(X_4 | Y_1 = y_1, \ldots, Y_4 = y_4) \).

(a) Show the complete work done by the elimination algorithm for the ordering \( X_2, X_1, X_3 \). Make sure you show the definition of all the factors introduced in the process, and what is returned at the end. What is the size of the largest factor generated during variable elimination?

Start by inserting evidence, which gives the following initial factors:

\[
p(X_1) p(X_2 | X_1) p(X_3 | X_2) p(X_4 | X_3) p(y_1 | X_1) p(y_2 | X_2) p(y_3 | X_3) p(y_4 | X_4)
\]

Eliminate \( X_2 \): \( f_1(X_1, X_3, y_2) = \sum_{X_2} p(X_2 | X_1) p(X_3 | X_2) p(y_2 | X_2) \), and get:

\[
f_1(X_1, X_3, y_2) p(X_1) p(X_4 | X_3) p(y_1 | X_1) p(y_3 | X_3) p(y_4 | X_4)
\]

Eliminate \( X_1 \): \( f_2(X_3, y_2, y_1) = \sum_{X_1} f_1(X_1, X_3, y_2) p(X_1) p(y_1 | X_1) \), and get:

\[
f_2(X_3, y_2, y_1) p(X_4 | X_3) p(y_3 | X_3) p(y_4 | X_4)
\]

Eliminate \( X_3 \): \( f_3(y_2, y_1, X_4, y_3) = \sum_{X_3} f_2(X_3, y_2, y_1) p(X_4 | X_3) p(y_3 | X_3) \), and get:

\[
f_3(y_2, y_1, X_4, y_3) p(y_4 | X_4)
\]

Finally, we have

\[
P(X_4, Y_1 = y_1, \ldots, Y_4 = y_4) = f_3(y_2, y_1, X_4, y_3) p(y_4 | X_4),
\]

so normalizing over \( X_4 \) gives \( P(X_4 | Y_1 = y_1, \ldots, Y_4 = y_4) \).

The largest factor is \( f_1(X_1, X_3, y_2) \), a 2-dimensional table.

(b) Show the complete work done by the elimination algorithm for the ordering \( X_1, X_2, X_3 \). Make sure you show the definition of all the factors introduced in the process, and what is returned at the end. What is the size of the largest factor generated during variable elimination?

Start by inserting evidence, which gives the following initial factors:

\[
p(X_1) p(X_2 | X_1) p(X_3 | X_2) p(X_4 | X_3) p(y_1 | X_1) p(y_2 | X_2) p(y_3 | X_3) p(y_4 | X_4)
\]

Eliminate \( X_1 \): \( f_1(X_2, y_1) = \sum_{X_1} p(X_1) p(X_2 | X_1) p(y_1 | X_1) \), and get:

\[
f_1(X_2, y_1) p(X_3 | X_2) p(X_4 | X_3) p(y_2 | X_2) p(y_3 | X_3) p(y_4 | X_4)
\]

Eliminate \( X_2 \): \( f_2(y_1, X_3, y_2) = \sum_{X_2} f_1(X_2, y_1) p(X_3 | X_2) p(y_2 | X_2) \), and get:

\[
f_2(y_1, X_3, y_2) p(X_4 | X_3) p(y_3 | X_3) p(y_4 | X_4)
\]
Eliminate $X_3$: $f_3(y_1, y_2, y_3, X_4) = \sum_{X_3} f_2(y_1, X_3, y_2)p(X_4|X_3)p(y_3|X_3)$, and get:

$$f_3(y_1, y_2, y_3, X_4)p(y_4|X_4)$$

Finally, we have

$$\mathbb{P}(X_4, Y_1 = y_1, \ldots, Y_4 = y_4) = f_3(y_1, y_2, y_3, X_4)p(y_4|X_4),$$

so normalizing over $X_4$ gives $\mathbb{P}(X_4|Y_1 = y_1, \ldots, Y_4 = y_4)$.

All factors are 1-dimensional, showing that the elimination order does matter!