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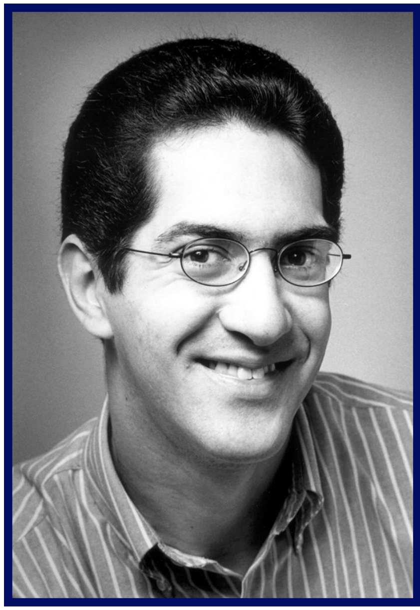
CS61C : Machine Structures

Lecture 2 – Number Representation

2004-09-01



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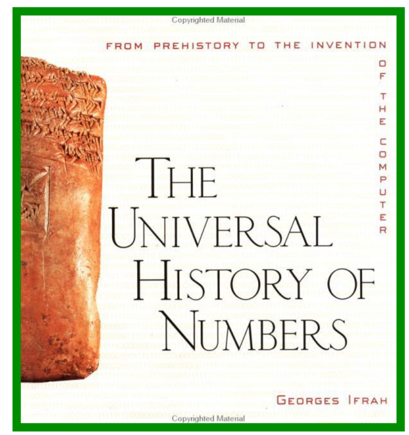
Great book ⇒
The Universal History of Numbers

by Georges Ifrah



Happy September 1st!

Decimal Numbers: Base 10

Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Example:

3271 =

$$(3 \text{x} 10^3) + (2 \text{x} 10^2) + (7 \text{x} 10^1) + (1 \text{x} 10^0)$$

Numbers: positional notation

- Number Base B ⇒ B symbols per digit:
  - Base 10 (Decimal): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
    Base 2 (Binary): 0, 1
- Number representation:
  - $d_{31}d_{30} \ldots d_1 d_0$ is a 32 digit number
  - value = $d_{31} \times B^{31} + d_{30} \times B^{30} + \ldots + d_1 \times B^1 + d_0 \times B^0$
- Binary: 0,1 (In binary digits called "bits")
  - 0b11010 $= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$
    $= 16 + 8 + 2$
    $= 26$
  - Here 5 digit binary # turns into a 2 digit decimal #
  - Can we find a base that converts to binary easily?

#s often written 0b…

Hexadecimal Numbers: Base 16

- **Hexadecimal:**
  0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F
  - Normal digits + 6 more from the alphabet
  - In C, written as 0x… (e.g., 0xFAB5)
- **Conversion: Binary⇔Hex**
  - 1 hex digit represents 16 decimal values
  - 4 binary digits represent 16 decimal values

  ⇒1 hex digit replaces 4 binary digits
- **One hex digit is a "nibble". Two is a "byte"**
- **Example:**
  - 1010 1100 0011 (binary) = 0x_____ ?

Decimal vs. Hexadecimal vs. Binary

Examples:

1010 1100 0011 (binary)
= 0xAC3

10111 (binary)
= 0001 0111 (binary)
= 0x17

0x3F9
= 11 1111 1001 (binary)

How do we convert between hex and Decimal?

00	0	0000
01	1	0001
02	2	0010
03	3	0011
04	4	0100
05	5	0101
06	6	0110
07	7	0111
08	8	1000
09	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

MEMORIZE!

What to do with representations of numbers?

- Just what we do with numbers!
  - Add them
  - Subtract them
  - Multiply them
  - Divide them
  - Compare them

```
      1   1
      1   0   1   0
+     0   1   1   1
-------------------------
  1   0   0   0   1
```

- Example: 10 + 7 = 17
  - …so simple to add in binary that we can build circuits to do it!
  - subtraction just as you would in decimal
  - Comparison: How do you tell if X > Y ?

Which base do we use?

- Decimal: great for humans, especially when doing arithmetic
- Hex: if human looking at long strings of binary numbers, its much easier to convert to hex and look 4 bits/symbol
  - Terrible for arithmetic on paper
- Binary: what computers use; you will learn how computers do +, -, *, /
  - To a computer, numbers always binary
  - Regardless of how number is written:

    $32_{ten} == 32_{10} == 0x20 == 100000_2 == 0b100000$
  - Use subscripts "ten", "hex", "two" in book, slides when might be confusing

BIG IDEA: Bits can represent anything!!

- Characters?
  - 26 letters $\Rightarrow$ 5 bits ($2^5 = 32$)
  - upper/lower case + punctuation
    $\Rightarrow$ 7 bits (in 8) ("ASCII")
  - standard code to cover all the world's languages $\Rightarrow$ 8,16,32 bits ("Unicode")
    `www.unicode.com`
- Logical values?
  - 0 $\Rightarrow$ False, 1 $\Rightarrow$ True
- colors ? Ex: *Red (00)* *Green (01)* *Blue (11)*
- locations / addresses? commands?
- MEMORIZE: N bits $\Rightarrow$ at most 2^N things

How to Represent Negative Numbers?

- So far, unsigned numbers
- Obvious solution: define leftmost bit to be sign!
  - 0 ⇒ +, 1 ⇒ -
  - Rest of bits can be numerical value of number
- Representation called sign and magnitude
- MIPS uses 32-bit integers. $+1_{ten}$ would be:

0000 0000 0000 0000 0000 0000 0000 0001

- And -1_{ten} in sign and magnitude would be:

1000 0000 0000 0000 0000 0000 0000 0001

Shortcomings of sign and magnitude?

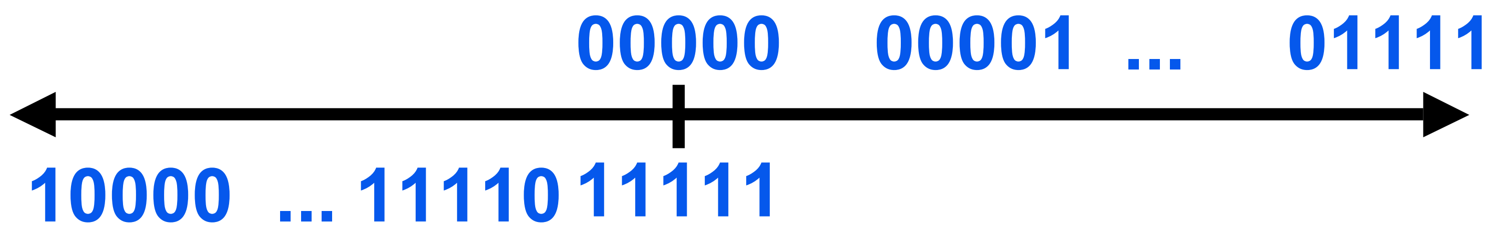
- Arithmetic circuit complicated
  - Special steps depending whether signs are the same or not
- Also, two zeros
  - $0x00000000 = +0_{ten}$
  - $0x80000000 = -0_{ten}$
  - What would two 0s mean for programming?
- Therefore sign and magnitude abandoned

Administrivia

- Firmware (n) : a set of computer instructions used so frequently that it is stored on a memory chip in a computer rather than being part of a program. [MS Word Dictionary]
- Look at class website often!
- Reading for next Wed: K&R Ch1-4
    - 1st Reading quiz due Wed [if not up by 6pm the day before, it's not due & will be postponed]
- Labs need to be finished in your lab session (& only your lab) unless extended by your TA
    - +1 bonus pt if you finish your lab in the first hour!
- Homework #1 up now, due Wed @ 11:59pm
- Homework #2 up soon, due following Wed

Another try: complement the bits

- Example: $7_{10} = 00111_2$ $-7_{10} = 11000_2$
- Called One's Complement
- Note: positive numbers have leading 0s, negative numbers have leadings 1s.



- What is -00000 ? Answer: 11111
- How many positive numbers in N bits?
- How many negative ones?

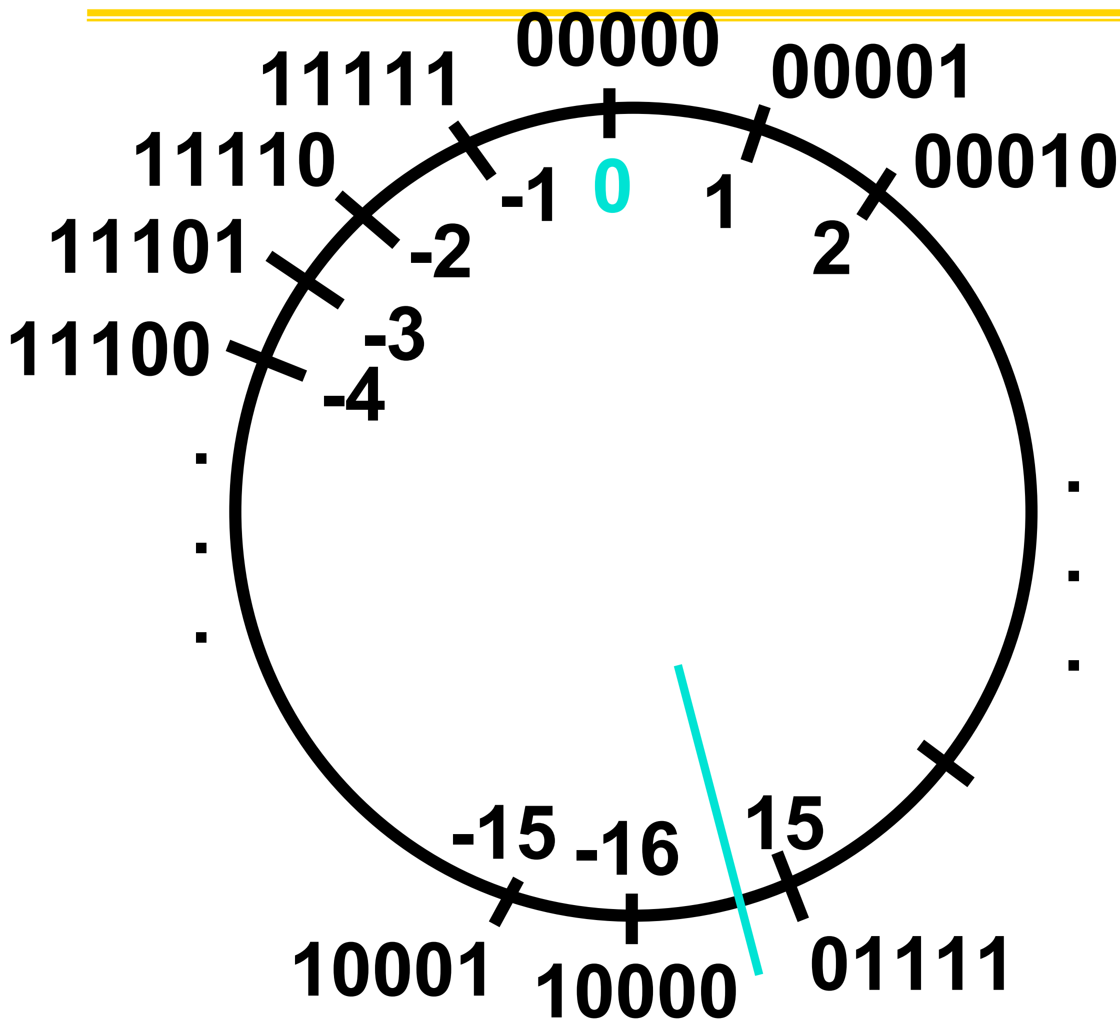
Shortcomings of One's complement?

- **Arithmetic still a somewhat complicated.**
- **Still two zeros**
  - `0x00000000` = $+0_{ten}$
  - `0xFFFFFFFF` = -0_{ten}
- **Although used for awhile on some computer products, one's complement was eventually abandoned because another solution was better.**

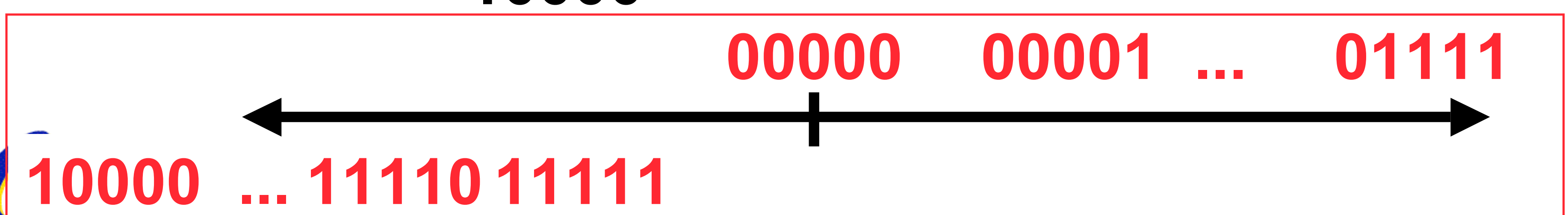
Standard Negative Number Representation

- **What is result for unsigned numbers if tried to subtract large number from a small one?**
  - **Would try to borrow from string of leading 0s, so result would have a string of leading 1s**
    - **3 - 4 ⇒ 00…0011 - 00…0100 = 11…1111**
  - **With no obvious better alternative, pick representation that made the hardware simple**
  - **As with sign and magnitude, leading 0s ⇒ positive, leading 1s ⇒ negative**
    - **000000...xxx is ≥ 0, 111111...xxx is < 0**
    - **except 1…1111 is -1, not -0 (as in sign & mag.)**
- **This representation is Two's Complement**

2’s Complement Number “line”: N = 5



- 2^{N-1} non-negatives
- 2^{N-1} negatives
- one zero
- how many positives?



Two's Complement for N=32

$0000 \ldots 0000\ 0000\ 0000\ 0000_{two}$	=		0_{ten}
$0000 \ldots 0000\ 0000\ 0000\ 0001_{two}$	=		1_{ten}
$0000 \ldots 0000\ 0000\ 0000\ 0010_{two}$	=		2_{ten}
. . .			
$0111 \ldots 1111\ 1111\ 1111\ 1101_{two}$	=		$2{,}147{,}483{,}645_{ten}$
$0111 \ldots 1111\ 1111\ 1111\ 1110_{two}$	=		$2{,}147{,}483{,}646_{ten}$
$0111 \ldots 1111\ 1111\ 1111\ 1111_{two}$	=		$2{,}147{,}483{,}647_{ten}$
$1000 \ldots 0000\ 0000\ 0000\ 0000_{two}$	=		$-2{,}147{,}483{,}648_{ten}$
$1000 \ldots 0000\ 0000\ 0000\ 0001_{two}$	=		$-2{,}147{,}483{,}647_{ten}$
$1000 \ldots 0000\ 0000\ 0000\ 0010_{two}$	=		$-2{,}147{,}483{,}646_{ten}$
. . .			
$1111 \ldots 1111\ 1111\ 1111\ 1101_{two}$	=		-3_{ten}
$1111 \ldots 1111\ 1111\ 1111\ 1110_{two}$	=		-2_{ten}
$1111 \ldots 1111\ 1111\ 1111\ 1111_{two}$	=		-1_{ten}

- One zero; 1st bit called sign bit
- 1 "extra" negative: no positive $2{,}147{,}483{,}648_{ten}$

Two's Complement Formula

- Can represent positive and negative numbers in terms of the bit value times a power of 2:

$$d_{31} \times -(2^{31}) + d_{30} \times 2^{30} + \ldots + d_2 \times 2^2 + d_1 \times 2^1 + d_0 \times 2^0$$

- Example: 1101_{two}

$= 1\text{x}-(2^3) + 1\text{x}2^2 + 0\text{x}2^1 + 1\text{x}2^0$

$= -2^3 + 2^2 + 0 + 2^0$

$= -8 + 4 + 0 + 1$

$= -8 + 5$

$= -3_{ten}$

Two's Complement shortcut: Negation

- Change every 0 to 1 and 1 to 0 (invert or complement), then add 1 to the result
- Proof: Sum of number and its (one's) complement must be $111...111_{two}$

  However, $111...111_{two} = -1_{ten}$

  Let x' $\Rightarrow$ one's complement representation of x

  Then $x + x' = -1 \Rightarrow x + x' + 1 = 0 \Rightarrow x' + 1 = -x$
- Example: -3 to +3 to -3

  x : $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two}$
  x': $0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two}$
  +1: $0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0011_{two}$
  ()': $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$
  +1: $1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two}$

You should be able to do this in your head…

Two's comp. shortcut: Sign extension

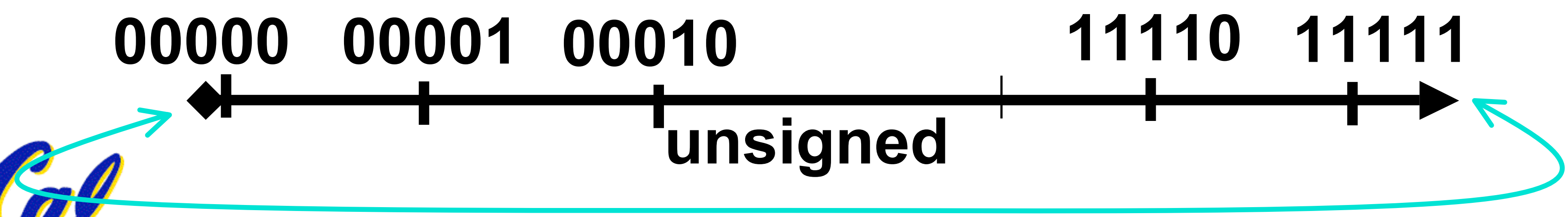
- Convert 2's complement number rep. using n bits to more than n bits
- Simply replicate the most significant bit (sign bit) of smaller to fill new bits
  - 2's comp. positive number has infinite 0s
  - 2's comp. negative number has infinite 1s
  - Binary representation hides leading bits; sign extension restores some of them
  - 16-bit -4_{ten} to 32-bit:

$$1111\ 1111\ 1111\ 1100_{two}$$

$$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$$

What if too big?

- Binary bit patterns above are simply representatives of numbers. Strictly speaking they are called "numerals".
- Numbers really have an ∞ number of digits
    - with almost all being same (00…0 or 11…1) except for a few of the rightmost digits
    - Just don't normally show leading digits
- If result of add (or -, *, /) cannot be represented by these rightmost HW bits, overflow is said to have occurred.



Peer Instruction Question

$X = 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$

$Y = 0011\ 1011\ 1001\ 1010\ 1000\ 1010\ 0000\ 0000_{two}$

A. X > Y (if signed)

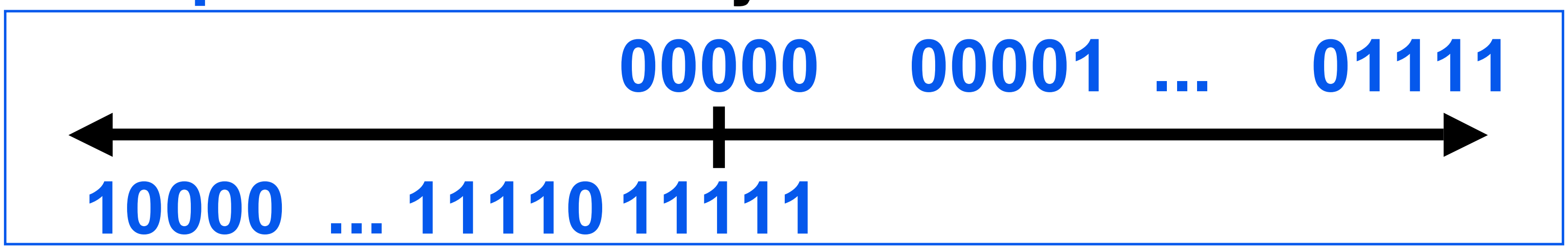
B. X > Y (if unsigned)

C. An encoding for Babylonians could have 2^N non-zero numbers w/N bits!

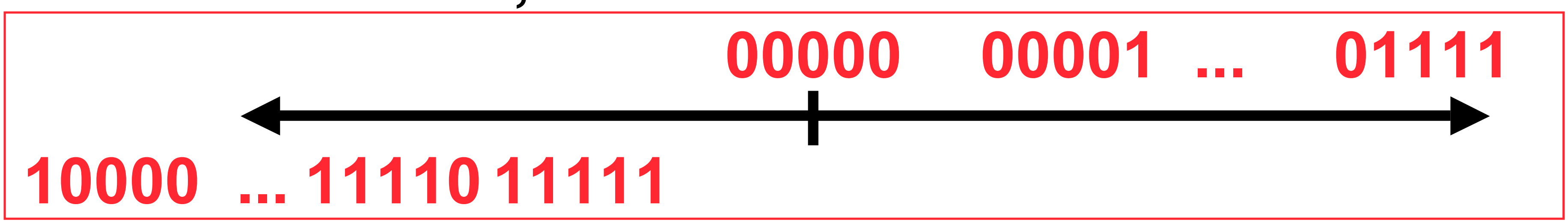
	ABC
1:	FFF
2:	FFT
3:	FTF
4:	FTT
5:	TFF
6:	TFT
7:	TTF
8:	TTT

And in Conclusion...

- We represent “things” in computers as particular bit patterns: N bits $\Rightarrow 2^N$
- Decimal for human calculations, binary for computers, hex to write binary more easily
- 1’s complement - mostly abandoned



- 2’s complement universal in computing: cannot avoid, so learn

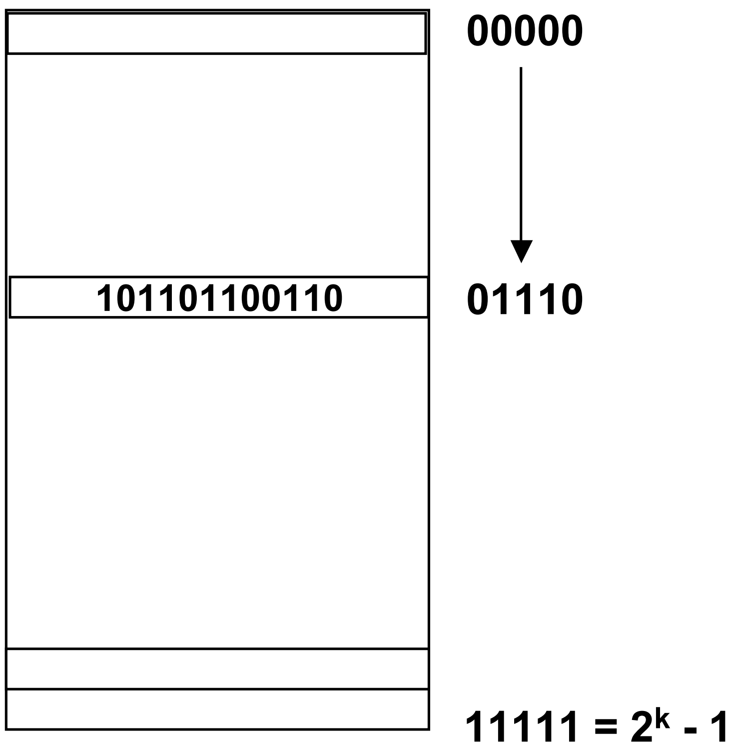


- Overflow: numbers ∞; computers finite, errors!

Bonus Slides

- Peer instruction let’s us skip example slides since you are expected to read book and lecture notes beforehand, but we include them for your review
- Slides shown in logical sequence order

BONUS: Numbers represented in memory



- Memory is a place to store bits
- A *word* is a fixed number of bits (eg, 32) at an address
- *Addresses* are naturally represented as unsigned numbers in C

BONUS: Signed vs. Unsigned Variables

- Java just declares integers `int`
    - Uses two's complement
- C has declaration `int` also
    - Declares variable as a signed integer
    - Uses two's complement
- Also, C declaration `unsigned int`
    - Declares a unsigned integer
    - Treats 32-bit number as unsigned integer, so most significant bit is part of the number, not a sign bit