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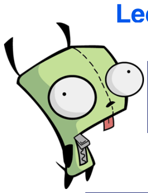
CS61C : Machine Structures

Lecture 15 – Floating Point I

2004-02-23



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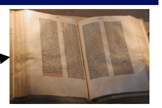

This day in history…

1455 - Publication of the Gutenberg Bible

1998 - Netscape founds Mozilla.org

Review of Numbers

- Computers are made to deal with numbers
- What can we represent in N bits?
  - Unsigned integers:

    0 to $2^N - 1$
  - Signed Integers (Two's Complement)

    $-2^{(N-1)}$ to $2^{(N-1)} - 1$

Other Numbers

- What about other numbers?
  - Very large numbers? (seconds/century)

    $3{,}155{,}760{,}000_{10}$ ($3.15576_{10} \times 10^9$)
  - Very small numbers? (atomic diameter)

    0.00000001_{10} ($1.0_{10} \times 10^{-8}$)
  - Rationals (repeating pattern)

    2/3 (0.666666666. . .)
  - Irrationals

    $2^{1/2}$ (1.414213562373. . .)
  - Transcendentals

    e (2.718...), π (3.141...)
- All represented in scientific notation

Scientific Notation (in Decimal)

$6.02_{10} \times 10^{23}$

mantissa → 6.02; exponent → 23; decimal point; radix (base) → 10

- Normalized form: no leadings 0s (exactly one digit to left of decimal point)
- Alternatives to representing 1/1,000,000,000
  - Normalized: 1.0×10^{-9}
  - Not normalized: 0.1×10^{-8}, 10.0×10^{-10}

Scientific Notation (in Binary)

$1.0_{two} \times 2^{-1}$

mantissa → 1.0; exponent → -1; "binary point"; radix (base) → 2

- Computer arithmetic that supports it called <u>floating point</u>, because it represents numbers where the binary point is not fixed, as it is for integers
  - Declare such variable in C as `float`

Floating Point Representation (1/2)

- Normal format: $+1.xxxxxxxxxx_{two} * 2^{yyyy}{}_{two}$
- Multiple of Word Size (32 bits)

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

- S represents Sign
  Exponent represents y's
  Significand represents x's
- Represent numbers as small as 2.0×10^{-38} to as large as 2.0×10^{38}

Floating Point Representation (2/2)

- What if result too large? (> 2.0×10^{38})
  - Overflow!
  - Overflow ⇒ Exponent larger than represented in 8-bit Exponent field
- What if result too small? (>0, < 2.0×10^{-38})
  - Underflow!
  - Underflow ⇒ Negative exponent larger than represented in 8-bit Exponent field
- How to reduce chances of overflow or underflow?

Double Precision Fl. Pt. Representation

- Next Multiple of Word Size (64 bits)

31	30 … 20	19 … 0
S	Exponent	Significand
1 bit	11 bits	20 bits

Significand (cont'd)
32 bits

- Double Precision (vs. Single Precision)
  - C variable declared as `double`
  - Represent numbers almost as small as 2.0×10^{-308} to almost as large as 2.0×10^{308}
  - But primary advantage is greater accuracy due to larger significand

QUAD Precision Fl. Pt. Representation

- Next Multiple of Word Size (128 bits)
- Unbelievable range of numbers
- Unbelievable precision (accuracy)
- This is currently being worked on
- The current version has 15 bits for the exponent and 112 bits for the significand
- Oct-Precision? That's just silly! It's been implemented before…

IEEE 754 Floating Point Standard (1/4)

- Single Precision, DP similar
- Sign bit: 1 means negative
  0 means positive
- Significand:
  - To pack more bits, leading 1 implicit for normalized numbers
  - 1 + 23 bits single, 1 + 52 bits double
  - always true: Significand < 1
    (for normalized numbers)
- Note: 0 has no leading 1, so reserve exponent value 0 just for number 0

IEEE 754 Floating Point Standard (2/4)

- Kahan wanted FP numbers to be used even if no FP hardware; e.g., sort records with FP numbers using integer compares
- Could break FP number into 3 parts: compare signs, then compare exponents, then compare significands
- Wanted it to be faster, single compare if possible, especially if positive numbers
- Then want order:
  - Highest order bit is sign (negative < positive)
  - Exponent next, so big exponent => bigger #
  - Significand last: exponents same => bigger #

IEEE 754 Floating Point Standard (3/4)

- Negative Exponent?
  - 2's comp? 1.0×2^{-1} v. $1.0 \times 2^{+1}$ (1/2 v. 2)

1/2	0	1111 1111	000 0000 0000 0000 0000 0000
2	0	0000 0001	000 0000 0000 0000 0000 0000

  - This notation using integer compare of 1/2 v. 2 makes 1/2 > 2!
- Instead, pick notation 0000 0001 is most negative, and 1111 1111 is most positive
  - 1.0×2^{-1} v. $1.0 \times 2^{+1}$ (1/2 v. 2)

1/2	0	0111 1110	000 0000 0000 0000 0000 0000
2	0	1000 0000	000 0000 0000 0000 0000 0000

IEEE 754 Floating Point Standard (4/4)

- Called Biased Notation, where bias is number subtract to get real number
  - IEEE 754 uses bias of 127 for single prec.
  - Subtract 127 from Exponent field to get actual value for exponent
  - 1023 is bias for double precision
- Summary (single precision):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

- $(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$
  - Double precision identical, except with exponent bias of 1023

"Father" of the Floating point standard

IEEE Standard 754 for Binary Floating-Point Arithmetic.

1989
ACM Turing
Award Winner!

Prof. Kahan

www.cs.berkeley.edu/~wkahan/
…/ieee754status/754story.html

Administrivia…Midterm in 2 weeks!

- Midterm 1 LeConte Mon 2004-03-07 @ 7-10pm
  - Conflicts/DSP? Email Head TA Andy, cc Dan
- How should we study for the midterm?
  - Form study groups -- don't prepare in isolation!
  - Attend the review session (2004-03-06 @ 2pm in 10 Evans)
  - Look over HW, Labs, Projects
  - Write up your 1-page study sheet--handwritten
  - Go over old exams – HKN office has put them online (link from 61C home page)

Upcoming Calendar

Week #	Mon	Wed	Thurs Lab	Fri
#6 This week	Holiday	Floating Pt I	Floating Pt	Floating Pt II
#7 Next week	MIPS inst. Format III	Running Program	Running Program	Running Program
#8 Midterm week	Digital Systems Midterm @ 7pm	State Elements	Finite State Machines	Comb. Logic Midterm grades out

Understanding the Significand (1/2)

- Method 1 (Fractions):
  - In decimal: $0.340_{10} \Rightarrow 340_{10}/1000_{10} \Rightarrow 34_{10}/100_{10}$
  - In binary: $0.110_2 \Rightarrow 110_2/1000_2 = 6_{10}/8_{10} \Rightarrow 11_2/100_2 = 3_{10}/4_{10}$
  - Advantage: less purely numerical, more thought oriented; this method usually helps people understand the meaning of the significand better

Understanding the Significand (2/2)

- Method 2 (Place Values):
  - Convert from scientific notation
  - In decimal: $1.6732 = (1 \times 10^0) + (6 \times 10^{-1}) + (7 \times 10^{-2}) + (3 \times 10^{-3}) + (2 \times 10^{-4})$
  - In binary: $1.1001 = (1 \times 2^0) + (1 \times 2^{-1}) + (0 \times 2^{-2}) + (0 \times 2^{-3}) + (1 \times 2^{-4})$
  - Interpretation of value in each position extends beyond the decimal/binary point
  - Advantage: good for quickly calculating significand value; use this method for translating FP numbers

Example: Converting Binary FP to Decimal

0	0110 1000	101 0101 0100 0011 0100 0010

- Sign: 0 => positive
- Exponent:
  - $0110\ 1000_{two} = 104_{ten}$
  - Bias adjustment: 104 - 127 = -23
- Significand:
  - $1 + 1x2^{-1} + 0x2^{-2} + 1x2^{-3} + 0x2^{-4} + 1x2^{-5} + ...$
    $= 1 + 2^{-1} + 2^{-3} + 2^{-5} + 2^{-7} + 2^{-9} + 2^{-14} + 2^{-15} + 2^{-17} + 2^{-22}$
    $= 1.0_{ten} + 0.666115_{ten}$
- Represents: $1.666115_{ten} * 2^{-23} \sim 1.986*10^{-7}$
  (about 2/10,000,000)

Converting Decimal to FP (1/3)

- Simple Case: If denominator is an exponent of 2 (2, 4, 8, 16, etc.), then it's easy.
- Show MIPS representation of -0.75
  - -0.75 = -3/4
  - $-11_{two}/100_{two} = -0.11_{two}$
  - Normalized to $-1.1_{two} \times 2^{-1}$
  - $(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$
  - $(-1)^1 \times (1 + .100\ 0000\ ...\ 0000) \times 2^{(126-127)}$

1	0111 1110	100 0000 0000 0000 0000 0000

Converting Decimal to FP (2/3)

- Not So Simple Case: If denominator is not an exponent of 2.
  - Then we can't represent number precisely, but that's why we have so many bits in significand: for precision
  - Once we have significand, normalizing a number to get the exponent is easy.
  - So how do we get the significand of a neverending number?

Converting Decimal to FP (3/3)

- Fact: All rational numbers have a repeating pattern when written out in decimal.
- Fact: This still applies in binary.
- To finish conversion:
  - Write out binary number with repeating pattern.
  - Cut it off after correct number of bits (different for single v. double precision).
  - Derive Sign, Exponent and Significand fields.

Peer Instruction

1	1000 0001	111 0000 0000 0000 0000 0000

What is the decimal equivalent of the floating pt # above?

```
1: -1.75
2: -3.5
3: -3.75
4: -7
5: -7.5
6: -15
7: -7 * 2^129
8: -129 * 2^7
```


"And in conclusion…"

- Floating Point numbers <u>approximate</u> values that we want to use.
- IEEE 754 Floating Point Standard is most widely accepted attempt to standardize interpretation of such numbers
  - Every desktop or server computer sold since ~1997 follows these conventions
- Summary (single precision):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

- $(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$
  - Double precision identical, bias of 1023