Due Friday, November 4, 5:00pm

You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper, and paper-clip (don't staple!) the sheets of paper together. Label each sheet of paper with your name, your class login, your discussion section number (101–108), and "CS70–Fall 2011". Turn in your homework and problem x into the box labeled "CS70 – Fall 2011, Problem x" whereon the 2nd floor of Soda Hall. Failure to follow these instructions will likely cause you to receive no credit at all.

1. (10 pts.) Geometric distribution

James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces temporary amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability $\frac{1}{3}$. On the average, how long does it take before he realizes that the door is unlocked and escapes?

Hint: if you're doing a lot of algebra, you're taking the wrong approach.

2. (24 pts.) Expectations

Solve each of the following problems using linearity of expectation. Clearly explain the steps. (Hint: for each problem, think about what the appropriate random variables should be and define them explicitly.)

(a) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "bonbon" appears?

(b) A coin with Heads probability p is flipped n times. A "run" is a maximal sequence of consecutive flips that are all the same. (Thus, for example, the sequence HTHHHTTH with $n = 8$ has five runs.) Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Justify your answer carefully.

3. (10 pts.) Misprints

A textbook has on average one misprint per page.

(a) What is the chance that you see exactly 4 misprints on page 1?

(b) What is the chance that you see exactly 4 misprints on some page in the textbook, if the textbook is 250 pages long?

[HINT: You may assume that misprints are "rare events" that obey the Poisson distribution.]

4. (24 pts.) Homework 9 Revisited

(a) In Q. 3 of Homework 9, we have computed the expectation of the number of packets lost under three different routing protocols, and they turn out to be all the same. Compute and compare the variances. Under which protocol is the variance the largest? the smallest?

(b) In Q. 5 of Homework 9, Alice decides to play the game against Bob because the net profit has a positive expectation. Now, suppose Alice is very risk averse and will only play the game if her chance of winning is greater than 95%.

(i) If she only has the option to play the game *once*, should she play?

(ii) Suppose she has the option to play the game n times, and the outcome each time is independent. Without doing any more calculations than what you already calculated in Q. 5 of Homework 9, can you conclude that there is always a large enough n such that Alice should play? Justify your answer.

(iii) If your answer to part (ii) is yes, then find an explicit lower bound on such an n.

5. (22 pts.) Higher Moments

Let X be the random variable corresponding to the final position of the n step drunken sailor walk discussed in class in lecture 25 (or the random walk discussed in notes 15 in the reader)

1. Use Chebyshev's inequality to give an upper bound on the probability that the sailor at least $t\sqrt{n}$ steps from the origin.
2. Compute $E[X^4]$ for this random variable.
   Hint: Use the fact that $X = X_1 + \cdots X_n$. When is $E[X_a X_b X_c X_d] \neq 0$?
3. Show that the $Pr[|X| \geq c] \leq \frac{E[X^4]}{c^4}$.
4. Use parts (2) and (3) to give an upper bound on the drunken sailor at least $t\sqrt{n}$ away from the origin. How does it compare with the result one gets from Chebyshev's inequality? For which values of t is it better? Is it better for $t = 1$? $t = 2$? $t = 100$?

6. (10 pts.) Total Annihilation

A superpower has 3450 missiles stored in well separated silos. An enemy is considering a sneak attack. However, for the attack to succeed every one of the missiles must be destroyed. Assume that each attacking warhead hits exactly one of the missile silos, with each silo being equally likely to be the one that is hit. How many warheads would you expect to be needed to ensure the complete destruction of every missile? Explain your answer; you may use any result from class without proof provided it is clearly stated.