CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao HW 12

Due Tuesday, November 22

You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper, and paper-clip (don't staple!) the sheets of paper together. Label each sheet of paper with your name, your discussion section number (101–108), your login, and "CS70–Fall 2011". Turn in your homework and problem x into the box labeled "CS70 – Fall 2011, Problem x" whereon the 2nd floor of Soda Hall. Failure to follow these instructions will likely cause you to receive no credit at all.

1. (12 pts.) Will They or Won't They?

Alice and Bob agree to try to meet for a dinner date between 6 and 7 PM at their favorite sushi restaurant. Being extremely busy (as both are near-constantly engaged in fanciful errands illustrating random CS problems), they are unable to specify their arrival times exactly. Specifically, we will model this by supposing that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, they each agree that if the other person is not there when they arrive, they will wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for their date?

2. (6 pts.) Your Favorite Subject: Questions About Grading Curves

The set of grades on a Discrete Math examination at an inferior school (not Berkeley[1]) are normally distributed with a mean of 74 and a standard deviation of 10. Find:

(a) the lowest passing grade if the bottom 5% of the students fail the class;

(b) the highest B if just the top 10% of the students are given A's.

NOTE: You may assume that if X is normal with mean 0 and variance 1, then $\Pr[X \leq 1.3] \approx 0.9$ and $\Pr[X \leq 1.65] \approx 0.95$.

3. (24 pts.) The Impersistence of Memory

We begin by proving two very useful properties of the exponential distribution. We then use them to solve a problem in digital photography.

(a) Let r.v. X have geometric distribution with parameter p. Show that, for any positive integers m, n, we have

$$\Pr[X > m+n \mid X > m] = \Pr[X > n].$$

NOTE: This is called the "memoryless" property of the geometric distribution, because it says that conditioning on the past does not change the future distribution.

[1] Not CMU either

(b) Let r.v. X have exponential distribution with parameter λ. Show that, for any positive s, t, we have

$$\Pr[X > s+t \mid X > t] = \Pr[X > s].$$

NOTE: This is the memoryless property of the exponential distribution.

(c) Let r.v.'s X_1, X_2 be independent and exponentially distributed with parameters λ_1, λ_2. Show that the r.v. $Y = \min\{X_1, X_2\}$ is exponentially distributed with parameter $\lambda_1 + \lambda_2$. [Hint: work with cdf 's.]

(d) You have a digital camera that requires two batteries to operate. You purchase n batteries, labeled $1, 2, \ldots, n$, each of which has a lifetime that is exponentially distributed with parameter λ and is independent of all the other batteries. Initially you install batteries 1 and 2. Each time a battery fails, you replace it with the lowest-numbered unused battery. At the end of this process you will be left with just one working battery. What is the expected total time until the end of the process? Justify your answer.[2]

4. (12 pts.) A Normal Problem

Let B be the number of Heads after flipping n coins, with Heads probability p; i.e., $B \sim \text{Binomial}(n, p)$. We have shown that $\mathbf{E}[B] = np$ and $\text{Var}(B) = np(1-p)$. It turns out that, for large n, the binomial distribution B approximates the normal distribution with the same mean and variance (this is a special case of the Central Limit Theorem). Use this information (and a suitable table or calculator for the cdf function of the normal distribution) to answer the following:

(a) Consider a coin which comes up heads 9 out of 10 times, on average. What is the (approximate) probability that, in 1000 independent flips of this coin, there will be at least 125 tails?

(b) Find a value k for which, when you flip a fair coin 10,000 times, the probability of k or more heads is approximately 0.20.

5. (24 pts.) A Convoluted Problem

(a) Let X and Y be two independent *discrete* random variables. Derive a formula for expressing the distribution of the sum $S = X+Y$ in terms of the distributions of X and of Y.

(b) Use your formula in part (a) to compute the distribution of $S = X+Y$ if X and Y are both discrete and uniformly distributed on $\{1, \ldots, K\}$.

(c) Suppose now X and Y are *continuous* random variables with densities f and g respectively (X, Y still independent). Based on part (a) and your understanding of continuous random variables, give an educated guess for the formula of the density of $S = X+Y$ in terms of f and g.

(d) Use your formula in part (c) to compute the density of S if X and Y have both uniform densities on $[0, a]$.

(e) Show that if X and Y are independent normally distributed variables, then $X+Y$ is also a normally distributed variable.

6. (12 pts.) Cumulative Final

A random variable is typically specified by an explicit *distribution* in the discrete case (that is, by a function describing the probability of each of its values) and by giving its *probability density function (pdf)* in the continuous case. We also introduced the idea of a *cumulative distribution function (cdf)* in the continuous case. We now show that cdfs are also applicable to discrete random variables.

[2] Alright, this isn't really a problem in digital photography. It's just a math problem phrased with the words "digital camera". Still, math is fun, right?

Recall that a cdf F is defined via $F(a) = \Pr[X \le a]$. [3]

(a) In the discrete case, show that the cdf F of a random variable X contains exactly the same information as the function defined via $G(a) = \Pr[X = a]$, by expressing F in terms of G and expressing G in terms of F.

(b) Compute and plot the cdf for (i) $X \sim \text{Geom}(p)$, (ii) $X \sim \text{Exp}(\lambda)$.

(c) Identify two key properties that a cdf of any r.v. has to satisfy.

[3] There is also an alternative convention under which cdfs are defined via $F(a) = \Pr[X < a]$. Either convention is acceptable for this problem, so long as you are clear about which you are using