CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao HW 13

Due Friday, December 2, 5:00pm

You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper, and paper-clip (don't staple!) the sheets of paper together. Label each sheet of paper with your name, your class login, your discussion section number (101–108), and "CS70–Fall 2011". Turn in your homework and problem x into the box labeled "CS70 – Fall 2011, Problem x" whereon the 2nd floor of Soda Hall. Failure to follow these instructions will likely cause you to receive no credit at all.

1. (15 pts.) How to Lie With Statistics

Here is some on-time arrival data for two airlines, A and B, into the airports of Los Angeles and Chicago. (Predictably, both airlines perform better in LA, which is subject to less flight congestion and less bad weather.)

	Airline A		Airline B	
	# flights	# on time	#flights	# on time
Los Angeles	600	534	200	188
Chicago	250	176	900	685

1. Which of the two airlines has a better chance of arriving on time into Los Angeles? What about Chicago?
2. Which of the two airlines has a better chance of arriving on time overall?
3. Do the results of parts (a) and (b) surprise you? Explain the apparent paradox, and interpret it by writing down precise expressions involving conditional probabilities.

2. (10 pts.) Functions

Prove that the composition of two bijections is a bijection. In other words, if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ (defined by $(g \circ f)(a) = g(f(a))$) is a bijection.

3. (54 pts.) To infinites, and beyond

Show whether each of the following sets is finite, countably infinite, or uncountable:

1. $\mathbb{N}$ (the set of all natural numbers)
2. $\mathbb{Z}$ (the set of all integers)
3. $\mathbb{Q}$ (the set of all rational numbers, i.e., numbers that can be expressed in the form a/b, where $a, b \in \mathbb{Z}$ and $b \neq 0$)

4. $\mathbb{R}$ (the set of all real numbers)
5. $\mathbb{C}$ (the set of all complex numbers)
6. $\{0,1\}^*$ (the set of all finite-length binary strings)
7. $\{0,1,2\}^*$ (the set of all finite-length ternary strings)
8. $\mathbb{Z}^3 = \{(a,b,c) : a,b,c \in \mathbb{Z}\}$ (the set of triples of integers)
9. $S = \{p(x) : p(x) = ax^2 + bx + c, \text{ where } a,b,c \in \mathbb{Z}\}$ (the set of all polynomials of degree at most 2 with integer coefficients)
10. $T = \{p(x) : p(x) = a_n x^n + \cdots + a_1 x + a_0, \text{ where } n \in \mathbb{N} \text{ and } a_0, a_1, \ldots, a_n \in \mathbb{Z}\}$ (the set of all polynomials with integer coefficients, of any degree)
11. $U = \{f : \mathbb{N} \to \{0,1\}\}$ (the set of all functions that map each natural number to 0 or 1)
12. $V = \{f : \mathbb{N} \to \mathbb{N}\}$ (the set of all functions that map each natural number to a natural number)
13. The set of all primes.
14. The set of all real-valued random variables on a finite sample space.
15. The set of all integer-valued random variables defined on the sample space Ω of positive integers, with $\Pr[\omega] = 1/2^\omega$
16. The set of all integer-valued random variables on a finite sample space.
17. The set of all possible functions from $\mathbb{Z}_{97}$ to $\mathbb{Z}_{97}$.
18. The set of all graphs.

4. (10 pts.) Diagonalization

Prove that the power set of the natural numbers, $P(\mathbb{N})$, the collection of subsets of $\mathbb{N}$ is uncountable. Hint: by contradiction assume there is $f : \mathbb{N} \to P(\mathbb{N})$. Consider $S = \{n \in \mathbb{N} | n \notin f(n)\}$.

Extra Problem

The next problem is for extra practice, and it will not be graded, so don't turn it in! We will provide solutions, however, and the material covered by this problem is included on the final, and you are responsible for it!

5. (0 pts.) Computability

We say that a computer program P computes the function $f : S \to \{0,1\}$ if, for every $x \in S$, when you run the program P on the input x, the program eventually finishes and returns the value $f(x)$. We say that a function f is *computable* if there exists some computer program that can compute f. (Programs always must be of finite length.)

Each of the following parts defines a function f. For each part, say whether f is computable or uncomputable. You don't need to justify your answer.

1. $f : \mathbb{N} \to \{0,1\}$, defined by

$$f(a) = \begin{cases} 1 & \text{if } a \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

2. $f : \mathbb{N} \to \{0,1\}$, defined by

$$f(a) = \begin{cases} 1 & \text{if there exist prime numbers } b,c \text{ such that } a = b + c, \\ 0 & \text{otherwise.} \end{cases}$$

3. $f : \mathbb{N} \to \{0,1\}$, defined by

$$f(a) = \begin{cases} 1 & \text{if Goldbach's conjecture is true,} \\ 0 & \text{otherwise.} \end{cases}$$

(Goldbach's conjecture is the following claim: for every even integer n greater than two, there exist two primes p, q such that $n = p + q$. No one knows whether the conjecture is true or false.)

4. $f : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$, defined by

$$f(P, I) = \begin{cases} 1 & \text{if program } P \text{ ever divides a number by zero at any point when run on input } I, \\ 0 & \text{otherwise.} \end{cases}$$