CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao HW 2

Due Friday, September 9, 5:00pm

You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper, and paper-clip (don't staple!) the sheets of paper together. Label each sheet of paper with your name, your discussion section number (101–108), and "CS70–Fall 2011". Turn in your homework and problem x into the box labeled "CS70 – Fall 2011, Problem x" whereon the 2nd floor of Soda Hall. Failure to follow these instructions will likely cause you to receive no credit at all.

1. (35 pts.) Practice Proving Propositions

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Lecture Note 2) you used.

1. For all natural numbers n, if n is even then $n^2 + 2011$ is odd.
2. For all natural numbers n, $n^2 + 5n + 1$ is odd.
3. For all real numbers a, b, if $a + b \geq 2011$ then $a > 1005$ or $b > 1005$.
4. For all real numbers r, if r is irrational then $r/4$ is irrational.
5. For all natural numbers n, $10n^2 > n!$.

2. (20 pts.) Interesting Induction

1. For $n \in N$ with $n \geq 2$, define s_n by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

   Prove that $s_n = 1/n$ for every natural number $n \geq 2$.
2. Let $a_n = 3^{n+2} + 4^{2n+1}$. Prove that 13 divides a_n for every $n \in N$. (Hint: What can you say about $a_{n+1} - 3a_n$?)

3. (28 pts.) Proofs, Perhaps

Which of the proofs below is correct? If a proof is incorrect, explain *clearly* and *concisely* where the logical error in the proof lies. (If the proof is correct, just mark it as correct – you don't need to give an explanation.) Simply saying that the claim (or induction hypothesis) is false is not enough!

1. **Claim:** $(\forall n \in \mathbf{N})(n^2 \leq n)$.
   Proof: Base Case: When $n = 1$, the statement is $1^2 \leq 1$ which is true.

Induction hypothesis: Assume that $k^2 \leq k$.

Inductive step: We need to show that

$$(k+1)^2 \leq k+1$$

Working backwards we see that:

$$k^2 \leq (k+1)^2 - 1 \leq (k+1) - 1 = k.$$

So we get back to our original hypothesis which is assumed to be true.
Hence, for every $n \in \mathbf{N}$ we know that $n^2 \leq n$. ♠

2. **Claim:** $(\forall n \in \mathbf{N})(7^n = 1)$.
Proof: (uses strong induction)
Base Case: Certainly $7^0 = 1$.

Induction hypothesis: Assume that $7^j = 1$ for all $0 \leq j \leq k$.

Inductive step: We need to prove that $7^{k+1} = 1$. But,

$$7^{k+1} = \frac{(7^k \cdot 7^k)}{7^{k-1}} = \frac{(1 \cdot 1)}{1} = 1$$

Hence, by the Principle of Strong Induction, $7^m = 1$ for all $m \in \mathbf{N}$. ♡

3. **Claim:** For all natural numbers $n \geq 4$, $2^n < n!$.
Proof: Base case: $2^4 = 16$ and $4! = 24$, so the statement is true for $n = 4$.
Inductive step: Assume that $2^n < n!$ for some $n \in \mathbf{N}$. Then $2^{n+1} = 2(2^n) < 2(n!) \leq (n+1)(n!) = (n+1)!$, so $2^{n+1} < (n+1)!$. By the principle of mathematical induction, the statement is true for all $n \geq 4$. ♣

4. **Claim:** All natural numbers are equal.
Proof: It is sufficient to show that for any two natural numbers a and b, $a = b$. Further, it is sufficient to show that for all $n \geq 0$, if a and b satisfy $\max\{a,b\} = n$ then $a = b$. We proceed by induction on n.
Base case: If $n = 0$ then a and b, being natural numbers, must both be 0. So clearly $a = b$.
Inductive step: Assume that the claim is true for some value n. Take a and b with $\max\{a,b\} = n+1$. Then $\max\{(a-1),(b-1)\} = n$, and hence by the induction hypothesis $(a-1) = (b-1)$. Consequently, $a = b$. ♢

4. (10 pts.) Take the Tokens

Tamara and Thuc are playing a game. They take turns removing tokens from a pile. Tamara moves first, and takes one, two or three tokens. If there are any tokens left, then it's Thuc's turn to take one, two or three tokens. They continue in the same way, and whoever takes the last token loses.

For example, if the pile starts with three tokens, than Tamara wins by taking two tokens in her first turn, forcing Thuc to take the last token. If the pile starts with five tokens, then no matter whether Tamara starts by taking one, two or three tokens, Thuc can win by taking all the rest of the tokens except one.

Prove that for all natural numbers k, if the pile starts with $4k+1$ tokens, then Thuc has a winning strategy.

5. (10 pts.) Rigorous Recursion Consider the following computer program:

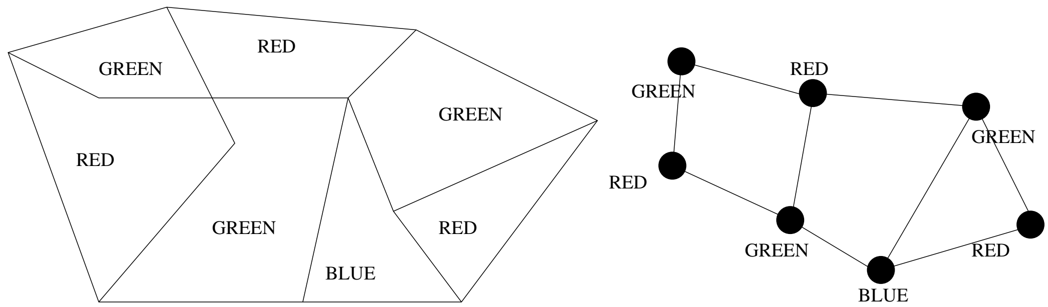

Figure 1: An island map that has been colored using three colors, and the corresponding graph.

function $G(n)$
  if $n = 0$ then return 0
  if $n = 1$ then return 1
  else return $5G(n-1) - 6G(n-2)$

Prove (using strong induction) that for all inputs $n \in \mathbf{N}$, the value returned by the program is $G(n) = 3^n - 2^n$.

6. (17 pts.) Coloring Countries

Explorers have just discovered several new islands in the Pacific ocean! Each island is divided into several countries. As chief map-maker, your job is to make a map of each island, giving a color to each country so that *no two neighboring countries have the same color*. For example, the left side of Figure 1 shows a map that has been colored using three colors.

Unfortunately, you haven't been to the mapmaking store in a while, and so you only have six colors to work with: red, green, blue, purple, orange and almond toast. Fortunately, that's enough to color any map, as we shall see!

The right side of Figure 1 shows another way of looking at a map: make a vertex for each country, and draw an edge between two nodes if the countries are neighbors. The graph you get will be *planar*, meaning it can be laid out so that none of the edges cross each other.

1. (2 pts.) In order to color the map in Figure 1 so that no neighbors have the same color, you need at least three different colors. (To see why, try to color it using only red and green and see what happens.) Draw a map that needs at least *four* different colors, and then draw the corresponding planar graph.
2. (15 pts.) In order to see that six colors will be enough to color any map, prove the following theorem:

   Theorem: *Every planar graph can be colored with six colors, in such a way that no two neighboring vertices have the same color.*

   You may find the following lemma useful:

   Lemma: *Every planar graph (with at least one node) has a node with at most five neighbors. (We say the node has* degree *at most five.)*

   You don't need to prove the lemma, but you may assume it is true when proving the theorem.

   (In fact, it only ever takes four colors to color a planar graph – but that's much harder to prove. Search for *four color theorem*.)