CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao HW 3

Due September 16, 5:00pm

Your answer to each question should be on its own sheet of paper, and **submit it to the drop box for that question** (i.e., your answer to question *i* goes into CS 70 drop box *i*). Print your name, discussion section, LOGIN, and question number on every sheet of paper.

1. (13 pts.) Strengthening the proposition

Prove that, for all $n \geq 1$, all entries of the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n$ are bounded above by n.

[HINT: Look at the title of this problem! To come up with a stronger hypothesis, try evaluating the matrix for the first few values of n.]

2. (18 pts.) Principle of induction

Let $P(k)$ be a proposition involving a natural number k. Suppose you know only that $(\forall k \in \mathbb{N})(P(k) \implies P(k+2))$ is true. For each of the following propositions, say whether the proposition is (i) definitely true, (ii) definitely false, or (iii) possibly (but not necessarily) true. Give a *brief* (one or two sentences) explanation for each of your answers.

1. $(\forall n \in \mathbb{N})(P(n))$.
2. $(\forall n \in \mathbb{N})(\neg P(n))$.
3. $P(0) \implies (\forall n \in \mathbb{N})(P(n+2))$.
4. $(P(0) \wedge P(1)) \implies (\forall n \in \mathbb{N})(P(n))$.
5. $(\forall n \in \mathbb{N})P(n) \implies ((\exists m \in \mathbb{N})(m > n + 2011 \wedge P(m)))$.
6. $(\forall n \in \mathbb{N})(n < 2011 \implies P(n)) \wedge (\forall n \in \mathbb{N})(n \geq 2011 \implies \neg P(n))$.

3. (10 pts.) Modular Arithmetic

(a) Give the addition and multiplication tables for modular-5 arithmetic. Write down the inverse for each of the elements which have one, and identify the ones which have no inverse.

(b) Solve the following equations for x and y or show that no solution exists. Show your work (in particular, what division must you carry out to solve each case).

(i) $5x + 23 \equiv 6 \mod 47$

(ii) $9x + 80 \equiv 2 \mod 81$

(iii) The system of simultaneous equations
$30x + 3y \equiv 0 \mod 37$ and $y \equiv 4 + 13x \mod 37$

(c) Compute gcd(5688,2010) and show your steps.

(d) Use Extended Euclid's algorithm to find some pair of integers j, k such that $52j + 15k = 3$. Show your work.

4. (15 pts.) Stable Marriage True-or-False? For each of the following claims, state whether the claim is true or false. If it is true, give a short proof; if it is false, give a simple counterexample.

(a) In a stable marriage instance, if man M and woman W each put each other at the top of their respective preference lists, then M must be paired with W in every stable pairing.

(b) In a stable marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists, then M cannot be paired with W in any stable pairing.

(c) For every n > 1, there is a stable marriage instance with n men and n women which has an unstable pairing in which every unmatched man-woman pair is a rogue couple.

5. (24 pts.) Stable Marriage

(a) Consider the following instance of the stable marriage problem:

Man	highest → lowest		
1	B	A	C
2	C	B	A
3	A	C	B

Table 1: Men's preference list

Woman	highest → lowest		
A	1	2	3
B	2	3	1
C	3	1	2

Table 2: Women's preference list

(i) List the rogue couples in the pairing $\{(1,C),(2,B),(3,A)\}$

(ii) List all the possible stable pairings (note that a single run of the algorithm from calss will only reveal one; there may be others).

(b) Run the "propose and reject" algorithm on the following example:

Man	highest → lowest			
1	B	C	A	D
2	C	A	B	D
3	A	B	C	D
4	B	C	A	D

Table 1: Men's preference list

Woman	highest → lowest			
A	4	1	2	3
B	2	3	4	1
C	3	4	1	2
D	1	2	3	4

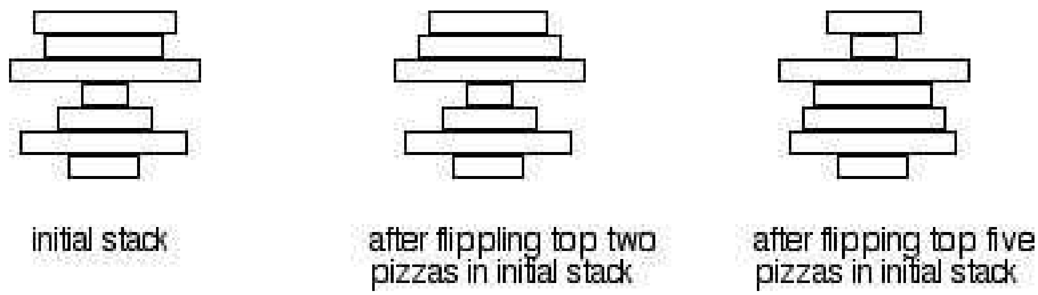
Table 2: Women's preference list

Using the same notation as in Note 4, page 30, illustrate the operation of your algorithm at each stage of the process. Show clearly the final stable pairing produced by your algorithm.

6. (20 pts.) The proof of the pi is in the eating

Dave is moonlighting as an intern at Emilia's Pizzeria (fantastic pizza! highly recommended!), where he is learning how to make great pizzas. However, the delightful smell of the cooking pizzas tends to make him a bit distracted.

One day, he has a stack of unbaked pizza doughs and for some unknown reason, he decides to arrange them in order of size, with the largest pizza on the bottom, the next largest pizza just above that, and so on. During his internship so far, he has learned how to place his spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). The figure below shows two sample flips.



This is the only move Dave can do to change the order of the stack; however, he is willing to keep repeating this kind of move until he gets the stack in order. Is it always possible for him to get the pizzas in order via some sequence of moves, no matter how many pizzas he starts with and no matter how they are arranged initially? Prove your answer.