Due Friday, October 7, 5:00pm

You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper, and paper-clip (don't staple!) the sheets of paper together. Label each sheet of paper with your name, your discussion section number (101–108), your login, and "CS70–Fall 2011". Turn in your homework and problem x into the box labeled "CS70 – Fall 2011, Problem x" whereon the 2nd floor of Soda Hall. Failure to follow these instructions will likely cause you to receive no credit at all.

1. (32 pts.) This problem counts more than the rest

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. You should leave your answers as (tidy) expressions involving factorials, binomial coefficients, etc., rather than evaluating them as decimal numbers (though you are welcome to perform this last step as well for your own interest if you like, provided it is clearly separated from your main answer). Also, you should explain clearly how you arrived at your answers; solutions provided without appropriate explanation will receive no credit.

1. How many 13-bit strings are there that contain exactly 5 ones?
2. How many 55-bit strings are there that contain more ones than zeros?
3. How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards within a bridge hand is considered irrelevant.)
4. How many different 13-card bridge hands contain no aces? [Recall that there are four aces in a standard 52-card deck]
5. How many different 13-card bridge hands contain all four aces?
6. How many different 13-card bridge hands contain exactly 5 spades? [Recall that there are 13 spades in a standard 52-card deck]
7. How many ways are there to order a standard 52-card deck?
8. Two identical decks of 52 distinct cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?
9. How many different anagrams of "KENTUCKY" are there? (An anagram of "KENTUCKY" is any re-ordering of the letters of "KENTUCKY"; i.e., any string made up of the eight letters K, E, N, T, U, C, K, and Y, in any order. The anagram does not have to be an English word.)
10. How many different anagrams of "ALASKA" are there?
11. How many different anagrams of "MINNESOTA" are there?
12. How many different anagrams of "MISSISSIPPI" are there?
13. Suppose you are given 8 distinguishable balls (numbered 1 through 8) and 24 bins. How many different ways are there to distribute these 8 balls among these 24 bins?

14. Suppose you are given 8 indistinguishable balls and 24 bins. How many distinguishable ways are there to distribute these 8 balls among these 24 bins?
15. Suppose you are given 8 indistinguishable balls and 5 bins. How many distinguishable ways are there to distribute these 8 balls among these 5 bins such that no bin is empty?
16. There are 30 students currently enrolled in a class. How many different ways are there to pair up the 30 students (that is, to split the class into groups of 2 students)?

2. (16 pts.) I've got another riddle for you

You have been hired as an actuary by a candy store, and have just been informed that an eccentric millionaire has distributed ten golden tickets to his newly re-opened chocolate factory among the 100 candy bars in your store (so there are 90 bars with no tickets, and 10 bars with one ticket each). Given that the store has twenty customers, each of whom will purchase five candy bars uniformly at random, calculate the probabilities for any particular customer of:

1. not receiving any tickets to the chocolate factory
2. receiving exactly one ticket to the chocolate factory
3. receiving at least one ticket to the chocolate factory

3. (20 pts.) Fermat's necklace

In the following, let p be a prime number and let k be a positive integer.

1. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable. We have a piece of string. We want to make a pretty decoration by threading p many beads onto the string (from left to right). We can choose any sequence of colors, subject only to one rule: the p beads must not all have the same color.

   How many different ways are there construct such a sequence of beads?

   (Your answer should be a simple function of k and p.)
2. Now we tie the two ends of the string together, forming a circular necklace. This lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors—red (R), green (G), and blue (B)—then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

   Count how many non-equivalent necklaces there are, if the p beads must not all have the same color. Again, your answer should be a simple function of k and p.

   [Hint: What can you conclude if rotating all the beads on a necklace to another position produces an identical looking necklace?]
3. How can you use the above to prove Fermat's little theorem? (Recall that Fermat's little theorem says that if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.)

4. (22 pts.) A good bet?

Your friend proposes the following game: Each round, she will roll six fair dice. If exactly four *distinct* numbers show up, then she will pay you a dollar. Otherwise, you will pay her a dollar. The game will go on for a thousand rounds. Would you be willing to play this game? Justify your answer with a calculation. [HINT: Be very careful in counting the number of ways that exactly four distinct numbers can show up! You should find that the game is rather finely balanced!]