CS 70 Discrete Mathematics and Probability Theory
Fall 2011 Rao HW 8

Due Friday, October 21, 5:00pm

You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper, and paper-clip (don't staple!) the sheets of paper together. Label each sheet of paper with your name, your class login, your discussion section number (101–108), and "CS70–Fall 2011". Turn in your homework and problem x into the box labeled "CS70 – Fall 2011, Problem x" whereon the 2nd floor of Soda Hall. Failure to follow these instructions will likely cause you to receive no credit at all.

1. (8 pts.) Marbles

Box A contains 1 black and 3 white marbles, and box B contains 2 black and 4 white marbles. A box is selected at random, and a marble is drawn at random from the selected box.

1. What is the probability that the marble is black?
2. Given that the marble is white, what is the probability that it came from the box A?

2. (20 pts.) Rehashing Note 12

Here is another way to (approximately) analyze the hashing problem discussed in Note 12. Recall that we are throwing m balls into n bins. For each pair (i, j) with $1 \leq i < j \leq m$, let B_{ij} denote the event that balls i and j land in the same bin.

1. What is $\Pr[B_{ij}]$?
2. Let B be the event that at least one collision occurs. (A *collision* happens when two balls land in the same bin.) Use the union bound together with part 1 to show that $\Pr[B] \leq \frac{m^2}{2n}$.
3. Deduce from part 2 the largest value you can choose for m, as a function of n, such that if m balls are thrown into n bins then the probability of a collision is at most $\frac{1}{2}$. How does your value compare with the value $m = 1.177\sqrt{n}$ we derived in class? Can you explain the difference?

3. (14 pts.) Find the Probabilities

Let A and B be two events. Suppose $\Pr[A] = 1/5$, $\Pr[A|B] = 2/5$ and $\Pr[B|A] = 1/2$.

1. What is $\Pr[B]$? Show your work.
2. What is $\Pr[A \cup B]$? Show your work.
3. Can A and B be independent events? Give a brief explanation.
4. Can A and B be disjoint events? Give a brief explanation.

4. (15 pts.) Count the Square-Frees

A positive integer is called *square-free* if it is not divisible by the square of any positive integer greater than 1. For example $35 = 5 \cdot 7$ is square-free but $18 = 2 \cdot 3^2$ is not. 1 is square-free. Use the principle of inclusion/exclusion to find the number of square-free positive integers strictly less than 201.

5. (16 pts.) Independence (due to H.W. Lenstra)

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

1. Prove or disprove: $\overline{A}$ and $\overline{B}$ are necessarily independent.
2. Prove or disprove: A and $\overline{B}$ are necessarily independent.
3. Prove or disprove: A and $\overline{A}$ are necessarily independent.
4. Prove or disprove: It is possible that $A = B$.

6. (12 pts.) Conditional Independence

On a particular day, let A be the event that Alekh has no homework, and let B be the event that the weather is nice. Assume $\Pr[A] = 1/3$ and $\Pr[B] = 2/3$, and that A and B are independent: $\Pr[A \cap B] = \Pr[A]\Pr[B]$.

1. If Alekh has no homework and the weather is nice, Alekh always plays volleyball, but otherwise (if Alekh has homework or the weather is not nice) Alekh never plays Volleyball. Let C be the event that Alekh plays volleyball. Express C in terms of A and B.
2. Conditioned on the event that Alekh plays volleyball, what is the probability that the weather is nice? Conditioned on the event that Alekh does *not* play volleyball, what is the probability that the weather is nice?
3. Here's a new definition: events D and E are *conditionally independent given F* if $\Pr[D \cap E | F] = \Pr[D|F]\Pr[E|F]$. Are A and B conditionally independent given C? Are they conditionally independent given $\overline{C}$?