Section 2

1. Let n be a composite number. Prove that there is a prime number $m \leq \sqrt{n}$ such that m divides n.

2. Let $x_0 = 1$ and $x_1, x_2, x_3 > 0$. Prove by contrapositive that, if $x_3 > 8$, then $\exists\, i \in \{0, 1, 2\}, \dfrac{x_{i+1}}{x_i} > 2$.

3. Prove the following by mathematical induction, where n is a positive integer:

(a.) $\displaystyle\sum_{i=1}^{n} 3i(i-1) = (n-1)n(n+1)$

(b.) $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

4. Prove that there are no positive integer solutions to the equation $x^2 = y^2 + 10$.

5. You are having fun shooting baskets in your backyard. You keep track of your success rate i.e. the fraction of your shots that have gone into the basket. In the morning, you found that your success rate was strictly worse than $\frac{2}{3}$. By lunch time, you found that your success rate was strictly greater than $\frac{2}{3}$. Prove that at some time in between, your success rate was exactly $\frac{2}{3}$.
[HINT: Prove by contradiction. If your success rate was never exactly $\frac{2}{3}$, there must have been a shot when it went from below $\frac{2}{3}$ to above $\frac{2}{3}$.]

6. Prove that for every positive integer n greater than 2, $3^n > n^2$.

7. Let m be an even natural number. Find natural numbers x and y such that $m = (x+y)^2 + 3x + y$. Example: for $m = 6$, $x = 0$ and $y = 2$ satisfy the equation. Try a few cases to find a pattern and then use induction to prove that the pattern works.

Hint: Let $m = 2n$, where $n \in \mathbb{N}$. Use induction to find x_n and y_n such that $2n = (x_n + y_n)^2 + 3x_n + y_n$.