Section 3

1. **[Induction]** Use induction to show that for any natural number $n \geq 1$, given pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ of integer numbers, there exist integer numbers c and d such that
   $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots (a_n^2 + b_n^2) = c^2 + d^2$.
   Hint: $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.

2. **[Strong Induction]** Use strong induction to prove that a class of $n \geq 12$ students can be broken into groups where each group has exactly 4 or 5 members.

3. **[Stable Marriage]** Assume we are given an instance to the stable marriage problem (i.e. a set of men, a set of women and their preferences) along with the following constraint. Each man has a list of women he finds unacceptable. That means that he would prefer to be left unmatched rather than being matched to any of the women he marks as unacceptable.

   (a) Give an algorithm for a stable marriage for this case (note that it may be possible now that not all men and women will have a match).

   (b) What if the women also have men they find unacceptable?

4. **[Stable Marriage]** In a group of n men and n women, Bob, one of the men, gets tipped off that he is the second-highest preference on every womans list. Bob is pretty happy to hear this. Assuming we use the traditional (male-optimal) algorithm, we can guarantee that at worst he will be matched the kth highest woman on his list for some $k \leq n$. What is k? Give a bad example where Bob is matched to the kth woman on his list.

5. **[Modular Arithmetic]**

   (a) What is the inverse of 5 modulo 7?

   (b) Do the following numbers have inverses modulo 3580225?

   $$5, \ 16, \ 29$$

   Give a short explanation for each.

6. **[Modular Arithmetic]**

   (a) Solve the following system of equations:

   $$5x \equiv 8y \pmod{13}$$

   $$x \equiv 9y - 11 \pmod{13}$$

   (b) Does the following equation have a solution?

   $$18x \equiv 19 \pmod{29}$$

   Prove your answer.

7. **[Modular Arithmetic]**

   (a) Let $F(n)$ denote the nth Fibonacci number. Show that for all $n \geq 1$, $\gcd(F(n+1), F(n)) = 1$.
   (Recall that the Fibonacci numbers are generated by the recursive relation $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$.)

   (b) Prove that an integer is divisible by 3 if and only if the sum of its digits in base 10 is divisible by 3.