Section 7

1. In the fictional state of West Dakota, license plates consist of 5 characters, each of which must be either a letter (one of the 26 from A through Z) or a digit (one of the 10 from 0 through 9). Beyond this, there are no further restrictions. How many possible West Dakotan license plates...

(a) contain only letters?

(b) have exactly three letters and two numbers?

(c) contain the string ABC?

(d) have at least two of the same character?

(Note: Order *does* matter; that is, two license plates with the same characters in a different order are considered distinct)

2. We wish to count the number of ways to throw k balls into n bins. We distinguish one final configuration from another only by the content of the bins, not by the order in which the balls land in the bins. The k balls and the n bins are *distinguishable*: each ball is labeled with a different number and each bin is given a distinct name.

(a) How many ways can we throw the balls into the bins such that the first and the second bins each get exactly two balls?

(b) If every possible way of throwing balls into bins is equally likely, what is the probability that the first and the second bins will each get exactly two balls?

(c) Now suppose $k = 2n$ and we would like to count the number of ways to throw the balls such that every bin gets exactly 2 balls. The following solution is proposed:

> We first place the k balls in a line. Then we throw the 1st and 2nd balls into the first bin, the 3rd and 4th balls into the second bin, etc. Thus, the number of ways to throw the balls is just the number of ways to arrange k distinct balls in a line, which is $k!$.

Explain why this 'solution' is wrong and give the correct answer.

(d) We can generalize the previous problem. Suppose we have $r_1 + r_2 + ...r_n = k$ where r_i are all natural numbers. If for all i we want the i-th bin to have r_i balls, how many ways are there to throw the balls?

3. In each of the following, decide which event is more likely. Assume a single die is equally likely to roll to 1, 2, 3, 4, 5 or 6.

(a) (i) Rolling at least one six when a die is rolled four times. (ii) Rolling a double six at least once when a pair of dice is rolled 24 times. (This problem was posed by the Chevalier te Méré and was solved by Blaise Pascale and Pierre de Fermat. *Hint:* $\left(\frac{35}{36}\right)^6 > \frac{5}{6}$.)

(b) (i) At least one number shows up at least twice when three dice are rolled. (ii) At least one number shows up at least twice when four dice are rolled.

4. (a) Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.
(*Hint:* count in two ways the number of ways to select a committee and a leader of the committee.)

(b) Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.
(*Hint:* count the number of ways to select a committee with n members, where the candidates are n politicians and n scientists, and then select a politician as the leader of the committee.)