Section 9

1. **An Important Distribution**
A drawer contains 10 socks, where 6 of them have holes and 4 of them do not. Suppose you pull two random socks out of the drawer, look at them, and then put them back. If you do this 5 times, what is the probability that you pull out a pair with no holes precisely 4 out of 5 times? What do you think is the distribution of the random variable describing the number of times you will pul out a pair with no holes if you repeat the experiment 5 times?

2. **Colorful Marketing Language**
A candy factory has an endless supply of red, orange and yellow jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each, with each possible combination of colors in the jar being equally likely. (One possible color combination, for example, is a jar of 56 red, 22 orange, and 22 yellow jelly beans.) As a marketing gimmick, the factory claims that the number of possible combinations is so large that there is negligible probability of finding two jars with the same color combination. You are skeptical about this claim and decide to do some calculations to test it.
   (a) Find n, the number of different possible color combinations of jelly beans in a single jar.
   (b) In terms of n, write down the probability that two jars of jelly beans have different color combinations.
   (c) Again in terms of n, write down the probability that m jars of jelly beans all have different color combinations. [NOTE: You do not need to simplify your expression.]
   (d) Approximately how many jars of jelly beans would you have to buy until the probability of seeing two jars with the same color combination is at least $\frac{1}{2}$? [NOTE: You should state your answer only as an order of magnitude (i.e., $10, 100, 1000, \ldots$). You may appeal to any result from class in order to derive your estimate; it should not be necessary to perform a detailed calculation.]

3. **Locked Out**
You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

   Find the distribution and the expectation of the number of trials you will need to open the front door, under the following alternative assumptions:
   (a) after an unsuccessful trial, you mark the corresponding key so that you never try it again, or
   (b) at each trial, you are equally likely to choose any key.

4. **Light Bulb**
I have a light bulb. On any given day, it has a probability p of burning out if it hasn't burnt out already. Let the random variable D represent the number of days until it burns out, so $Pr[D = 0] = p$, $Pr[D = 1] = p(1-p)$ and so on.
   (a) If n is a nonnegative integer, what is $Pr[D = n]$ as a simple function of n and p?
   (b) If n is a positive integer, what is $Pr[D > n]$ as a simple function of n and p?
   (c) Prove that if X is any random variable that takes values in $\mathbb{N}$, then $E[X] = \sum_{k=0}^{\infty} Pr[X > k]$
   (d) Calculate the expected time $E[D]$ until the light bulb burns out as a simple function of p.

5. **Another Important Distribution**
Suppose you are at a casino and betting on a game in which you have probability p of winning in each round. Your strategy is to bet k in the first round and then in each subsequent round, double the amount of money you bet in the previous round. So you would bet k, $2k$, $4k$, $8k$, and so on. You stop as soon as you earn a profit.

(a) Suppose you have unlimited money. What is the expected amount of money you will earn?

(b) What is the expected number of rounds you will play?

6. Yet Another Important Distribution

Professor Rao is trying to hitchhike along a deserted road. The probability that a car drives by during any given minute is 5%.

(a) Let A_i be the event that a car drives by in the i^{th} minute. The events $A_1, \ldots, A_{20}$ are mutually independent. What's the probability that no cars will appear in a particular span of 20 minutes?

(b) Now suppose Professor Rao has waited 20 minutes and no cars have come. What's the probability that no cars will arrive during the next 20 minutes?