# CS 70 Discrete Mathematics and Probability Theory
# Summer 2012 Filiba HW 2

## Due Thursday, June 28, 4:59pm

You may work in groups of up to 4 people (no larger!). Please read the group collaboration policies on the course website at `http://inst.cs.berkeley.edu/~cs70/su12/` before beginning group work. You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper. Label each sheet of paper with your name, student ID, section number (101, 102, or 103), the assignment number, the problem number, and "CS70–Summer 2012." Turn in your homework in the boxes labeled "CS70" on the 2nd floor of Soda Hall. Submit each problem separately in its appropriate box (i.e., your answer to question *i* goes into CS 70 drop box *i*). Failure to follow these instructions may cost you points, or cause you to receive no credit at all.

**Note that this homework is due at 4:59pm on Thursday.**

**1. (12 pts.) Grade these answers**

You be the grader. Students have submitted the following answers to several exam questions. Assign each student answer either an A (correct yes/no answer, valid justification), a D (correct yes/no answer, invalid justification), or an F (incorrect answer). As always, $\pi = 3.14159\ldots$

(a) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
**Student answer:** Yes. $2\pi = 6.283\ldots$, which is less than 100. Also $\pi = 3.1459\ldots$ is less than 50. Therefore the proposition is of the form True $\implies$ True, which is true.

(b) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 50$. Explain your answer.
**Student answer:** Yes. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$.

(c) **Exam question:** Is the following proposition true? $2\pi < 100 \implies \pi < 49$. Explain your answer.
**Student answer:** No. If $2\pi < 100$, then dividing both sides by two, we see that $\pi < 50$, which does not imply $\pi < 49$.

(d) **Exam question:** Is the following proposition true? $\pi^2 < 5 \implies \pi < 5$. Explain your answer.
**Student answer:** No, it is false. $\pi^2 = 9.87\ldots$, which is not less than 5, so the premise is false. You can't start from a faulty premise.

**2. (20 pts.) Practice with proofs**
Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

(a) For all natural numbers $n$, if $n$ is odd then $n^2 + 2010$ is odd.

(b) For all natural numbers $n$, $n^2 + 5n + 1$ is odd.

(c) For all real numbers $a, b$, if $a + b \geq 2010$ then $a \geq 1005$ or $b \geq 1005$.

(d) For all real numbers $r$, if $r$ is irrational then $3r$ is irrational.

(e) For all natural numbers $n$, $10n^2 > n!$.

**3. (20 pts.) Interesting Induction**

(a) For $n \in \mathbb{N}$ with $n \geq 2$, define $s_n$ by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

(b) Let $a_n = 3^{n+2} + 4^{2n+1}$. Prove that 13 divides $a_n$ for every $n \in N$. (Hint: What can you say about $a_{n+1} - 3a_n$?)

**4. (28 pts.) Proofs, Perhaps**

Which of the proofs below is correct? If a proof is incorrect, explain *clearly* and *concisely* where the logical error in the proof lies. (If the proof is correct, just mark it as correct – you don't need to give an explanation.) Simply saying that the claim (or induction hypothesis) is false is not enough!

(a) **Claim:** $(\forall n \in \mathbb{N})(n^2 \leq n)$.

**Proof:** Base Case: When $n = 1$, the statement is $1^2 \leq 1$ which is true.

Induction hypothesis: Assume that $k^2 \leq k$.

Inductive step: We need to show that

$$(k+1)^2 \leq k+1$$

Working backwards we see that:

$$k^2 \leq (k+1)^2 - 1 \leq (k+1) - 1 = k.$$

So we get back to our original hypothesis which is assumed to be true.
Hence, for every $n \in \mathbb{N}$ we know that $n^2 \leq n$. ♠

(b) **Claim:** $(\forall n \in \mathbb{N})(7^n = 1)$.

**Proof:** (uses strong induction)

Base Case: Certainly $7^0 = 1$.

Induction hypothesis: Assume that $7^j = 1$ for all $0 \leq j \leq k$.

Inductive step: We need to prove that $7^{k+1} = 1$. But,

$$7^{k+1} = \frac{(7^k \cdot 7^k)}{7^{k-1}} = \frac{(1 \cdot 1)}{1} = 1$$

Hence, by the Principle of Strong Induction, $7^m = 1$ for all $m \in \mathbf{N}$. ♡

(c) **Claim:** For all natural numbers $n \geq 4$, $2^n < n!$.

**Proof:** Base case: $2^4 = 16$ and $4! = 24$, so the statement is true for $n = 4$.

Inductive step: Assume that $2^n < n!$ for some $n \in \mathbb{N}$. Then $2^{n+1} = 2(2^n) < 2(n!) \leq (n+1)(n!) = (n+1)!$, so $2^{n+1} < (n+1)!$. By the principle of mathematical induction, the statement is true for all $n \geq 4$. ♣

(d) **Claim:** All natural numbers are equal.

**Proof:** It is sufficient to show that for any two natural numbers $a$ and $b$, $a = b$. Further, it is sufficient to show that for all $n \geq 0$, if $a$ and $b$ satisfy $\max\{a,b\} = n$ then $a = b$. We proceed by induction on $n$.

Base case: If $n = 0$ then $a$ and $b$, being natural numbers, must both be 0. So clearly $a = b$.

Inductive step: Assume that the claim is true for some value $n$. Take $a$ and $b$ with $\max\{a,b\} = n+1$. Then $\max\{(a-1),(b-1)\} = n$, and hence by the induction hypothesis $(a-1) = (b-1)$. Consequently, $a = b$. $\diamondsuit$

**5. (18 pts.) Principle of induction**

Let $P(k)$ be a proposition involving a natural number $k$. Suppose you know only that $(\forall k \in \mathbb{N})(P(k) \implies P(k+2))$ is true. For each of the following propositions, say whether the proposition is (i) definitely true, (ii) definitely false, or (iii) possibly (but not necessarily) true. Give a *brief* (one or two sentences) explanation for each of your answers.

(a) $(\forall n \in \mathbb{N})(P(n))$.

(b) $(\forall n \in \mathbb{N})(\neg P(n))$.

(c) $P(0) \implies (\forall n \in \mathbb{N})(P(n+2))$.

(d) $(P(0) \wedge P(1)) \implies (\forall n \in \mathbb{N})(P(n))$.

(e) $(\forall n \in \mathbb{N})(P(n) \implies ((\exists m \in \mathbb{N})(m > n+2010 \wedge P(m))))$.

(f) $(\forall n \in \mathbb{N})(n < 2010 \implies P(n)) \wedge (\forall n \in \mathbb{N})(n \geq 2010 \implies \neg P(n))$.

**6. (10 pts.) Recurrence relations**

Let $f(n)$ be defined by the recurrence relation $f(n) = 7f(n-1) - 10f(n-2)$ (for all $n \geq 2$) and $f(0) = 1$, $f(1) = 2$. Prove that $f(n) = 2^n$ for every $n \in \mathbb{N}$.

**7. (10 pts.) Rigorous Recursion** Consider the following computer program:

function $G(n)$
  if $n = 0$ then return 0
  if $n = 1$ then return 1
  else return $5G(n-1) - 6G(n-2)$

Prove (using strong induction) that for all inputs $n \in \mathbb{N}$, the value returned by the program is $G(n) = 3^n - 2^n$.

**8. (15 pts.) Let's be social**

$n$ people go to a bar. Initially, each person sits at their own table. After a little while, the bartender picks a table, taps the person at that table on the shoulder, and asks him to move to a second table. The person who just moved introduces himself and shakes hands with the person who was already sitting at the second table.

In general, the bartender keeps repeating the following operation: the bartender chooses two tables; the bartender asks everyone sitting at the first table to move over to the second table; and each of the folks who just moved from the first table shake hands with everyone who was already sitting at the second table. Suppose that there were $k$ people sitting at the first table and $\ell$ people sitting at the second table before this operation. After this operation, there are 0 people at the first table and $k+\ell$ people at the new table. Also, each of the $k$ newcomers shakes hands with each of the $\ell$ folks already at the second table, so $k\ell$ handshakes occur during this operation. The bartender repeats this kind of operation until all $n$ people are sitting at the same table.

Let $H(n)$ denote the the total number of handshakes that have occurred among the $n$ people by the time this process is finished and everyone is seated at the same table. Prove that it doesn't matter what order the bartender decides to choose tables; we always have $H(n) = n(n-1)/2$.

**9. (20 pts.) Stable marriage**

(a) Consider the following instance of the stable marriage problem:

| Man | highest→lowest | | |
|---|---|---|---|
| 1 | *B* | *A* | *C* |
| 2 | *C* | *B* | *A* |
| 3 | *A* | *C* | *B* |

Table 1: Men's preference list

| Woman | highest→lowest | | |
|---|---|---|---|
| *A* | 1 | 2 | 3 |
| *B* | 2 | 3 | 1 |
| *C* | 3 | 1 | 2 |

Table 2: Women's preference list

(i) List the rogue couples in the pairing $\{(1,C),(2,B),(3,A)\}$.

(ii) List **all** the possible stable pairings (note that a single run of the algorithm from class will only reveal one; there may be others).

(b) Run the "propose and reject" algorithm on the following example:

| Man | highest→lowest | | | |
|---|---|---|---|---|
| 1 | *B* | *C* | *A* | *D* |
| 2 | *C* | *A* | *B* | *D* |
| 3 | *A* | *B* | *C* | *D* |
| 4 | *B* | *C* | *A* | *D* |

Table 3: Men's preference list

| Woman | highest→lowest | | | |
|---|---|---|---|---|
| *A* | 4 | 1 | 2 | 3 |
| *B* | 2 | 3 | 4 | 1 |
| *C* | 3 | 4 | 1 | 2 |
| *D* | 1 | 2 | 3 | 4 |

Table 4: Women's preference list

Use the same notation as in Note 4, page 25, to illustrate the operation of your algorithm at each stage of the process. Show clearly the final stable pairing produced by your algorithm.