CS 70 Discrete Mathematics and Probability Theory
Summer 2012 Filiba HW 4

Due Thursday, July 12, 4:59pm

1. (32 pts.) Counting, counting, and more counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many 10-bit strings are there that contain exactly 4 ones?

(b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

(c) How many different 13-card bridge hands are there that contain no aces?

(d) How many different 13-card bridge hands are there that contain all four aces?

(e) How many different 13-card bridge hands are there that contain exactly 6 spades?

(f) How many 99-bit strings are there that contain more ones than zeros?

(g) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

(h) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(i) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

(j) How many different anagrams of ALASKA are there?

(k) How many different anagrams of ALABAMA are there?

(l) How many different anagrams of MONTANA are there?

(m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(o) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(p) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

2. (12 pts.) Counting telephone numbers

For the purposes of this problem, a phone number is an arbitrary sequence of 7 decimal digits.

(a) A *single-digit number* is a 7-digit phone number made up out of exactly one number. (For instance, 888-8888, 000-0000, and 555-5555 are single-digit numbers.) How many different single-digit numbers are there?

(b) A *non-repetitious number* is a 7-digit phone number where no digit is used more than once. (For instance, 571-2834, 102-9543, and 019-6273 are non-repetitious numbers, but 523-3678 is not.) How many different non-repetitious numbers are there?

(c) A *taxicab number* is a 7-digit phone number made up out of exactly two different digits. (For instance, 888-5858, 626-6666, 525-5252, 511-5115, and 000-1001 are taxicab numbers. 718-7818 and 777-7777 are not taxicab numbers.) How many different taxicab numbers are there?

3. (10 pts.) Algebraic vs. combinatorial proofs

Consider the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) Prove the identity by algebraic manipulation (using the formula for the binomial coefficients).

(b) Prove the identity using a combinatorial argument.

4. (14 pts.) Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 4 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a Heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a Heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first flip and the third flip are both Heads. List all the outcomes in C. How many are there?

(e) Let D be the event that the first flip or the third flip is a Heads. List all the outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express the event C in terms of A and B. Express the event D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 4 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

5. (8 pts.) Odd man out

There are 99 students enrolled in CS70. How many ways are there to pair them up into 2 student teams, with 1 left over?

6. (10 pts.) Even point in

Let $\{(x_i, y_i) : i = 1, 2, 3, 4, 5\}$ be a set of five distinct points in the plane with integer coordinates. Show that the midpoint of the line segment joining at least one pair of these points has integer coordinates.

7. (10 pts.) I know a guy who knows a guy

Prove that in a class with at least two students, at least two students know the same number of other students in the class.