CS 70 Discrete Mathematics and Probability Theory
Summer 2012 Filiba HW 5

Due Wednesday, July 18, 4:59pm

You may work in groups of up to 4 people (no larger!). Please read the group collaboration policies on the course website at `http://inst.cs.berkeley.edu/~cs70/su12/` before beginning group work. You *must* write up the solution set entirely on your own. You must never look at any other students' solutions (not even a draft), nor share your own solutions (not even a draft).

Please put your answer to each problem on its own sheet of paper. Label each sheet of paper with your name, student ID, section number (101, 102, or 103), the assignment number, the problem number, and "CS70–Summer 2012." Turn in your homework in the boxes labeled "CS70" on the 2nd floor of Soda Hall. Submit each problem separately in its appropriate box (i.e., your answer to question i goes into CS 70 drop box i). Failure to follow these instructions may cost you points, or cause you to receive no credit at all.

Note that this homework is due at 4:59pm on Wednesday.

1. (12 pts.) Probability Models
Suppose you have two coins, one is biased with a probability of p coming up Heads, and one is biased with a probability of q coming up Heads. Answer the questions below, but you don't need to provide justifications.

(a) Suppose $p = 1$ and $q = 0$.

(i) You pick one of the two coins randomly and flip it. You repeat this process n times, each time randomly picking one of the two coins and then flipping it. Consider the sample space Ω of all possible length n sequences of Heads and Tails so generated. Give a reasonable probability assignment (i.e. assign probabilities to all the outcomes) to model the situation.

(ii) Now you pick one of the two coins randomly, but flip the *same* coin n times. Identify the sample space for this experiment together with a reasonable probability assignment to model the situation. Is your answer the same as in the previous part?

(b) Repeat the above two questions for arbitrary values of p and q. Express your answers in terms of p and q.

2. (10 pts.) Playing with a ball
Alice, Bob and Chuck are playing with a ball. If Alice has the ball, she throws it to Chuck. If Bob has the ball, he throws it to Alice or Chuck, with equal probabilities. If Chuck has the ball, he throws it to Alice or Bob, with equal probabilities. At the beginning of the game the ball is given to one of Alice, Bob and Chuck, with equal probabilities. What is the probability that, after the ball is thrown once, Alice has it? That Bob has it? That Chuck has it?

3. (10 pts.) Independent Events
Let S be a sample space, and A and B be two independent events in S. Let $\bar{A} = S \backslash A$ and $\bar{B} = S \backslash B$. Prove that $\bar{A}$ and $\bar{B}$ are also independent. Are A and $\bar{B}$ independent? Are A and $\bar{A}$ independent?

4. (10 pts.) A fun game
Consider a game in which you have two quarters and a table with a row of squares marked like this:

2	3	4	5	6	7	8	9	10	11	12

Before the game begins, you get to place each quarter on one square. You can put either both quarters on the same square, or you can put them on two different squares: your choice.

Then, you roll two fair dice, sum up the numbers showing on the dice to get a number from 2–12, and if there's a quarter on the square labelled with that number, remove it from the table. (If there are two quarters on that square, remove only one of them.) Now roll the two fair dice a second time, again getting a number from 2–12, and again removing a single quarter from the square with that number, if there's a quarter there. At this point, the game is over. If you removed both quarters, you win; if any quarter remains on the table, you lose.

(a) What's the probability of winning, if you put two quarters on the square labelled 5?

(b) What's your best strategy? In other words, what's the best place to put your two quarters, if you want to maximize the probability of winning? State where you should put your two quarters. Then, calculate the probability that you win, if you put your two quarters there.

Be careful! This one is a little tricky. You don't have to prove your answer correct on your homework solution, but you might want to do it on scratch paper for your own sake anyway.

5. (10 pts.) Conditional probability

(a) I have a bag containing either a \$1 or \$5 bill (with probability $1/2$ for each of these two possibilities). I then add a \$1 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag (without looking). Suppose it turns out to be a \$1 bill. If a second student draws the remaining bill from the bag, what is the chance that it too is a \$1 bill? Show your calculations.

(b) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work.

6. (12 pts.) Beat Watson

You've been booked to play a Jeopardy tournament where to win the tournament, you have to win two *consecutive* games of Jeopardy, out of three games. You have the choice of playing Andy, then Watson, then Andy (AWA)—or Watson, then Andy, then Watson (WAW). Andy is a lousy Jeopardy player; Watson is very difficult to beat. Which schedule should you choose, to maximize your chances of winning the tournament?

(If you win the first and third game of Jeopardy but lose the second, you *lose* the tournament!) Let's analyze this puzzle. Suppose you have probability p of beating Watson in any given game, and probability q of beating Andy in any single game. Assume all three games are independent, draws never happen, and $0 < p < q < 1$.

(a) What is the probability that you win the tournament, if you choose the AWA schedule?

(b) What is the probability that you win the tournament, if you choose the WAW schedule?

(c) Which schedule offers you the better chances of winning the tournament? Does your answer depend upon the specific values of p, q? Prove your answer.

7. (10 pts.) Pairwise vs mutual independence

In this problem we will see that pairwise independence of events does not imply mutual independence.

Two fair dice are thrown. Let A be the event that the number on the first die is odd, B the event that the number on the second die is odd, and C the event that the sum of the two numbers is odd.

(a) Show that the three events A,B,C are *pairwise independent* (i.e., each pair (A,B), (B,C) and (A,C) is independent).

(b) Show that the events A,B,C are *not* mutually independent.

8. (10 pts.) Discreet math

A non-profit wants to poll a sample of people to ask them whether they have ever had an extramarital affair. This being an extremely sensitive subject, one obvious problem is that if the surveyers ask this question straight-out, respondents may lie to avoid revealing personal information about their private lives.

The surveyers come up with the following clever scheme. They will ask the respondent to secretly roll a fair die. If the die comes up 1, 2, 3, or 4, the respondent is supposed to answer truthfully. If the die comes up 5 or 6, the respondent is supposed to answer the *opposite* of the truthful answer. The respondent is cautioned not to reveal what number came up on the die. Notice that if the respondent answers "Yes," this answer is not necessarily incriminating: for all the surveyer knows, this particular respondent might have rolled a 5 or 6 and might have never had an affair in his/her life.

Let p be the probability that, if we select a person at random, then they will have had an extramarital affair. (Of course, the surveyers do not know p; that is what they want to estimate.) Let q denote the probability that, if we select a person at random and have them follow the scheme above, then they will answer "Yes."

(a) Calculate a simple formula for q, as a function of p.

(b) Next, suppose the surveyers have estimated q. Now they want to solve for p. Find a simple formula for p, as a function of q.

9. (12 pts.) Practice with Bayes rule

In 1998, researchers at the Max Planck Institute conducted a survey of 24 doctors (doctors who had an average of 14 years of professional experience), to study how they understand risk. The doctors were presented with the following information:

> The probability that a woman has breast cancer is 0.008. If she has breast cancer, the probability that a mammogram will show a positive result is 0.90. If a woman does not have breast cancer the probability of a positive result is 0.07. Take, for example, a woman who has a positive result. What is the probability that she actually has breast cancer?

Each doctor was asked to estimate the probability that a woman whose mammogram comes back positive actually has breast cancer. One-third of the doctors said "0.90"; another one-third's estimates were in the range 0.50–0.80; one-sixth estimated something in the range 0.05–0.10; and another one-sixth estimated about 0.01.

(a) Calculate the correct probability that a woman has breast cancer, given that her mammogram shows a positive result. Show your work. Evaluate your answer to get a concrete number.

(b) What fraction of doctors were in the right ballpark? Which ones?

(c) For the doctors whose answers weren't close, how would you explain to them why the number you got in part (a) is correct, in terms they'd likely be able to understand? The doctors probably don't need to know how to get the exact answer, but they need to understand why your answer is approximately right.

(If your explanation involves mathematical formulas, references to Bayes theorem and conditional probability, or technical jargon, many doctors are probably going to have a hard time understanding what you're trying to say. How could you explain this in terms that an intelligent layperson would have a fighting chance of understanding?)